MATH 400

Real Analysis

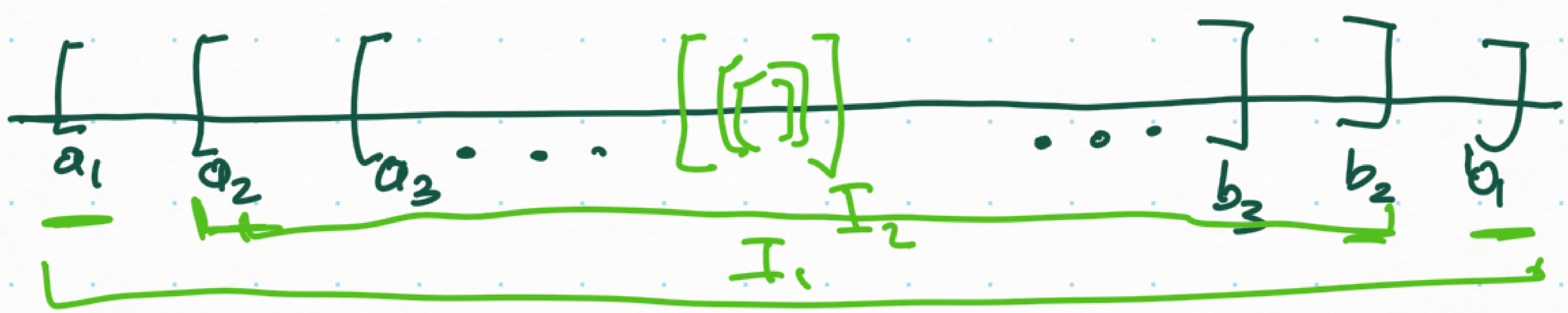
Part #5

Nested Interval Property

For each $n \in \mathbb{N}$, $I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \leq x \leq b_n\}$ is a given closed interval s.t. $I_n \supseteq I_{n+1}$ $\forall n = 1, 2, 3, \ldots\ldots$

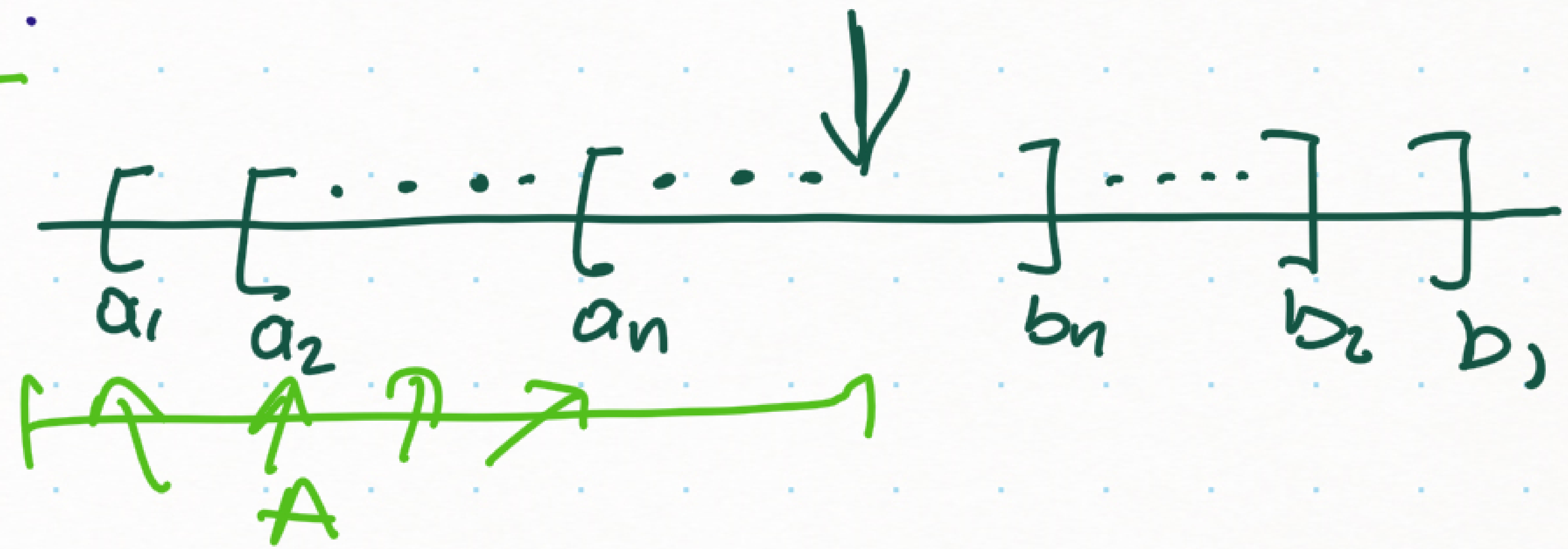
Then, this nested sequence of closed intervals has a nonempty intersection: $\boxed{\bigcap_{n=1}^{\infty} I_n \neq \emptyset}$

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots\ldots$$



Proof Idea is to use AOC (Axiom of completeness) to show $\exists\, a \in I_n \ \forall n$.

$$A = \{a_n \mid n \in \mathbb{N}\}$$



Is A bounded above?
yes, all $b_i \geq a_i \ \forall i$

Let $a = \sup A$ by AoC

Does $a \in I_n = [a_n, b_n]$?

a is an upper bound of A, i.e. $a \geq a_i \ \forall i$

$\Rightarrow a \geq a_n$

a is the least upper bd. of A.
& each b_n is an upper bd. of A) $\Rightarrow a \leq b_n$

$\Rightarrow a \in I_n = [a_n, b_n]$ for every n $\Rightarrow a \in \bigcap_{n=1}^{\infty} I_n$ ∎

Archimedean Property [$\mathbb{N}$ is unbounded]

① Let $x \in \mathbb{R}$, then $\exists\, n \in \mathbb{N}$ s.t. $n > x$

② Let $y > 0$ in $\mathbb{R}$, then $\exists\, n \in \mathbb{N}$ s.t. $1/n < y$

Proof ① Assume $\mathbb{N}$ is bounded above.
Since $\mathbb{N}$ nonempty subset of $\mathbb{R}$, by AoC, $\mathbb{N}$ must have a sup. Let $\alpha = \sup \mathbb{N}$

Consider $\alpha - 1$ ← Is this an up. bd. of $\mathbb{N}$? No

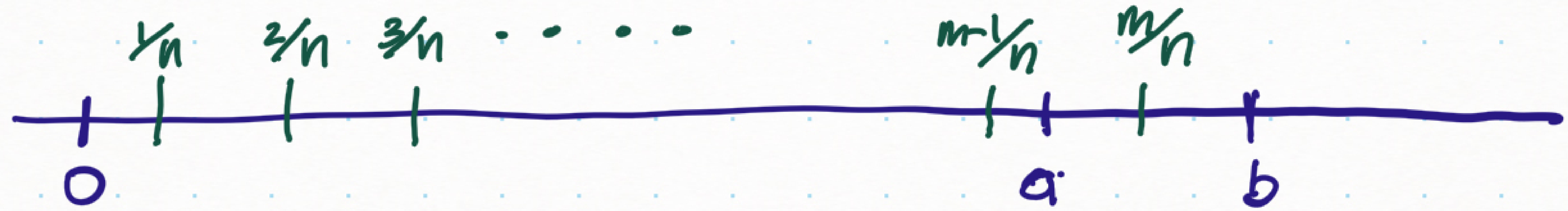
$\exists\, n \in \mathbb{N}$ s.t. $\alpha - 1 < n \Rightarrow \underbrace{n+1}_{\in \mathbb{N}} > \alpha \Rightarrow$ α cannot be an u.b. of $\mathbb{N}$.

② Apply ① with $x = 1/y$.
$y > 0 \Rightarrow \frac{1}{y} \in \mathbb{R} \underset{①}{\Rightarrow} \exists\, n > \frac{1}{y} \Rightarrow y > \frac{1}{n}$. contradiction.

Theorem [$\mathbb{Q}$ is dense in $\mathbb{R}$]

For every two real numbers $a<b$, $\exists\, r \in \mathbb{Q}$ s.t. $a<r<b$.

Proof We want to find $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ s.t. $a < \frac{m}{n} < b$.



By Archimedean Prop., we can pick $n \in \mathbb{N}$ s.t.

$$\frac{1}{n} < b-a$$

we know.

$$na < m < nb.$$

We want to choose m s.t. $m > na$

let m be the smallest integer larger than na

i.e., $m-1 \le na < m$

$\Rightarrow a < \frac{m}{n}$ ✓

e.g. $na = 2.73$

$m = 3$, $2 \le 2.73 < 3$

$na = 3$

$m = 4$, $3 \le 3 < 4$

we know $a < b - \frac{1}{n}$ $\left(\because \frac{1}{n} < b-a\right)$

$\Rightarrow m \le na+1 < n\left(b - \frac{1}{n}\right) + 1 = nb - 1 + 1 = nb$

$\therefore a < \frac{m}{n} < b$ $\left(\because a < \frac{m}{n} \text{ \& } m < nb\right)$ ∎

<u>Corollary</u> Given any two real numbers $a < b$, $\exists\ t \in \mathbb{I}$ (irrational number) s.t. $a < t < b$

<u>Proof</u> Apply above theorem ($\mathbb{Q}$ dense in $\mathbb{R}$) to $a - \sqrt{2}$ and $b - \sqrt{2}$ $\in \mathbb{R}$.

Think!

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Real Analysis

Part #6

Theorem There exists $\alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$.

Proof [**Recall** the example we did in part #4.

$S = \{r \in \mathbb{Q} \mid r^2 < 2\}$ S was bdd. above But it didn't have a least upper bound in $\mathbb{Q}$]

Let $T = \{t \in \mathbb{R} \mid t^2 < 2\}$

set $\alpha = \sup T$ (by AoC)

Claim ① $\alpha^2 < 2$ is not possible

② $\alpha^2 > 2$ is not possible

<u>Claim</u> $\alpha^2 < 2$ is not possible

<u>Assume $\alpha^2 < 2$</u>

[Why not? α is not an upper bound of T

Find $t \in T$ s.t. $\alpha < t$]

$$\left(\alpha + \frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2}$$

want < 2

$$< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} = \alpha^2 + \frac{2\alpha+1}{n}$$

[Since $\alpha^2 < 2$ we need to make $\boxed{\frac{2\alpha+1}{n}}$ small enough so that $\alpha^2 + \frac{2\alpha+1}{n} < 2$]

Choose $n_0 \in \mathbb{N}$ s.t. $\alpha^2 + \frac{2\alpha+1}{n_0} < 2$

i.e., $\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1}$ [possible by Archimedean Property

$\therefore \left(\alpha + \frac{1}{n_0}\right)^2 < \alpha^2 + \frac{2\alpha+1}{n_0} < 2$

So, $\alpha + \frac{1}{n_0} \in T$ but $\alpha + \frac{1}{n_0} > \alpha$, an upper bd. of T

contradiction.

Claim $\alpha^2 > 2$ is not possible

Assume $\alpha^2 > 2$

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2}$$
$$> \alpha^2 - \frac{2\alpha}{n}$$

Why not?
α is not the least upper bd. of T.
Find an u. bound b of T s.t. $b < \alpha$

Repeat the idea from previous claim & show that $\alpha - \frac{1}{n_0}$ is an upper bound of T (which is smaller than α).

$\boxed{}$

Ques How would you adapt this proof to show $(x)^{1/m}$ exists for any $x \geq 0$ & $m \in \mathbb{N}$?

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Real Analysis

Part # 7

How do we count the number of elements in a set?

cardinality of the set

Why are $\{0, 1, 2, 3, \dots\}$ the counting numbers?

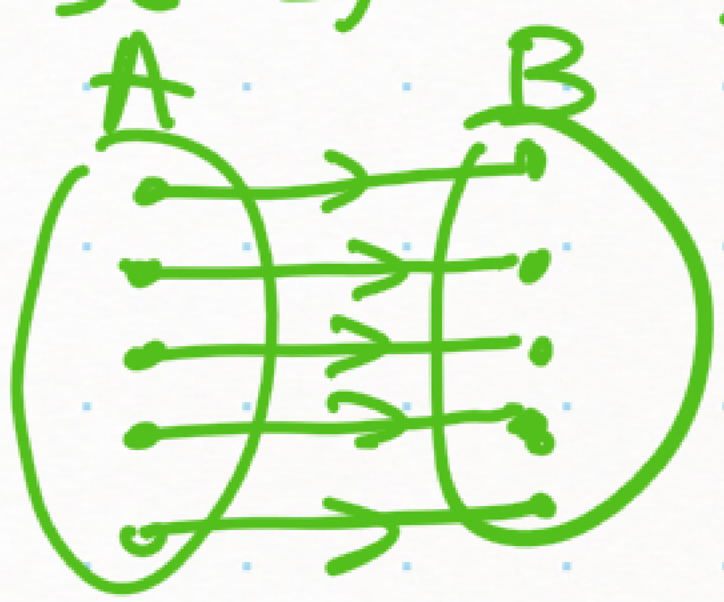
Why do we say there are 3 birds in this?

Established a bijection with $\{1, 2, 3\}$

What happens when a set has infinitely many elements?

Georg Cantor (1845-1918)

Defn A function $f: A \to B$ is a bijection if it is one-to-one ($a_1 \neq a_2$ in $A \Rightarrow f(a_1) \neq f(a_2)$ in B) and onto ($\forall b \in B, \exists a \in A$ s.t. $f(a) = b$)



Defn We say $A \sim B$, A has same cardinality as B if $\exists$ $f: A \to B$ that is a bijection.

Hilbert's Hotel

Hilbert's hotel has infinitely many rooms in a row.
A Friday evening, the hotel is full (all rooms are occupied)

One new guest arrives →

Two new guests arrive →

Infinitely many new guests arrive →

Hilbert's Hotel

Hilbert's hotel has infinitely many rooms in a row.

A Friday evening, the hotel is full (all rooms are occupied)

One new guest arrives → move every guest from their current room to the next room over.

Now, room #1 is empty.

Two new guests arrive → ✓

Infinitely many new guests arrive →

Room #1 → Room #2

Room #2 → Room #4

⋮

Room #n → Room #2n

⋮

All odd numbered rooms are now empty.

Example Let $E = \{2, 4, 6, \cdots\}$ set of evens

$E \subsetneq \mathbb{N}$ but $E \sim \mathbb{N}$

$$\begin{array}{cccccccccc}
\mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n & \cdots \\
& \updownarrow & \updownarrow & \updownarrow & \updownarrow & & & & \updownarrow & \\
E: & 2 & 4 & 6 & 8 & 10 & 12 & \cdots & 2n & \cdots
\end{array}$$

$f : \mathbb{N} \to E$ s.t. $f(n) = 2n$ $\forall n \in \mathbb{N}$

Check f is a bijection

Observation Writing all of a set B as a list with no repetitions Then this gives us a bijection between $\mathbb{N}$ & B.

$B : b_1 \; b_2 \; b_3 \; b_4 \; \cdots\cdots$

Example Let $E = \{2, 4, 6, \dots\}$ set of evens

$E \subsetneq \mathbb{N}$ but $E \sim \mathbb{N}$. $f : \mathbb{N} \to E$ as $f(n) = 2n \ \forall n$.

$$
\begin{array}{cccccccccc}
\mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow & \\
E: & 2 & 4 & 6 & 8 & 10 & 12 & \cdots & 2n & \cdots
\end{array}
$$

Example $\mathbb{N} \subsetneq \mathbb{Z}$ but $\mathbb{N} \sim \mathbb{Z}$

$f : \mathbb{N} \to \mathbb{Z}$

$$
f(n) = \begin{cases} (n-1)/2 & \text{if } n \text{ odd} \\ -n/2 & \text{if } n \text{ even} \end{cases}
$$

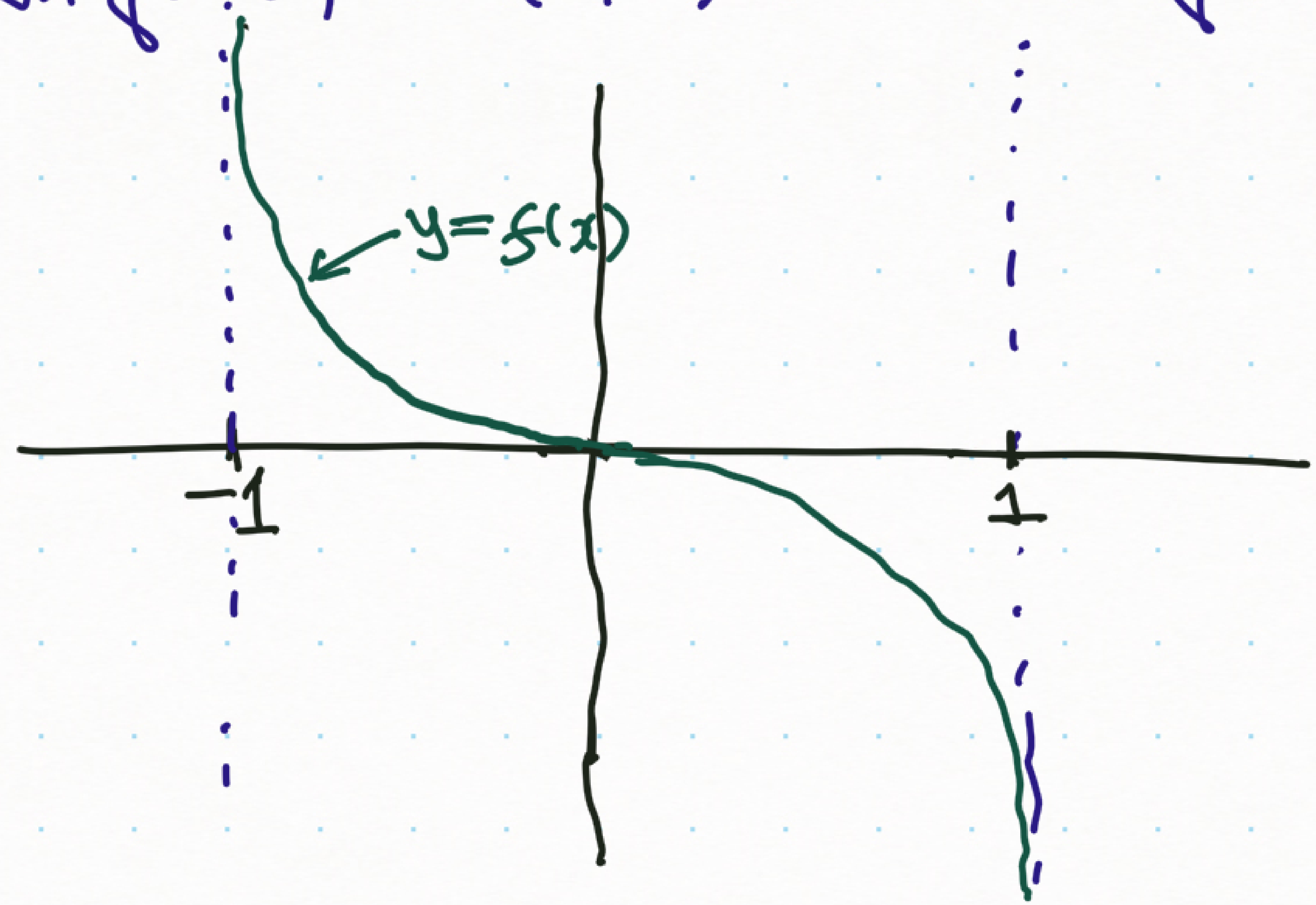
$$
\begin{array}{ccccccccc}
\mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots & \cdots & \\
\mathbb{Z}: & 0 & -1 & 1 & -2 & 2 & -3 & 3 & \cdots
\end{array}
$$

Example Show that $f(x) = \dfrac{x}{x^2-1}$ gives a bijection between the interval $(-1, 1)$ and $\mathbb{R}$

So, $(-1, 1) \sim \mathbb{R}$

In fact, $(a, b) \sim \mathbb{R}$ for any interval (a, b)

graph of f



Defn Set A is <u>countable</u> if $\mathbb{N} \sim A$.

An infinite set that is not countable is called an <u>uncountable set</u>.

Finite sets
Countable sets
Uncountable sets

<u>Are there any uncountable sets?</u> $\cancel{\mathbb{N}}$ $\cancel{\mathbb{Z}}$ $\mathbb{Q}$? $\mathbb{R}$?

Defn Set A is <u>countable</u> if $\mathbb{N} \sim A$.

An infinite set that is not countable is called an <u>uncountable set</u>.

<u>Are there any uncountable sets?</u> $\cancel{\mathbb{N}}$ $\cancel{\mathbb{Z}}$ $\cancel{\mathbb{Q}}$ $\mathbb{R}$

Theorem $\mathbb{Q}$ is countable

(Even though there are infinitely many rationals between any two integers!!

Proof.

$$
\begin{array}{cccccccccccccc}
\mathbb{N} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\
\mathbb{Q} & \underbrace{0} & \underbrace{1/1 \quad -1/1} & & \underbrace{1/2 \quad -1/2 \quad 2/1 \quad -2/1} & & & & \underbrace{\frac{1}{3} \quad -\frac{1}{3} \quad \frac{3}{1} \quad -\frac{3}{1}} & & & & \cdots
\end{array}
$$

Let think of $\mathbb{Q}^+$ first

	1	2	3	4	5	6	7	...
1	$1/1$	$2/1$	$3/1$	$4/1$	$5/1$	$6/1$	$7/1$	...
2	$1/2$	$2/2$	$3/2$	$4/2$	$5/2$	$6/2$	$7/2$	...
3	$1/3$	$2/3$	$3/3$	$4/3$	$5/3$	$6/3$	$7/3$	...
4	$1/4$	$2/4$	$3/4$	$4/4$	$5/4$	$6/4$	$7/4$	...
5	$1/5$	$2/5$	$3/5$	$4/5$	$5/5$	$6/5$	$7/5$	...
6	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	$6/6$	$7/6$	.....
7	$1/7$	$2/7$	$3/7$	$4/7$	$5/7$	$6/7$	$7/7$	...
⋮								

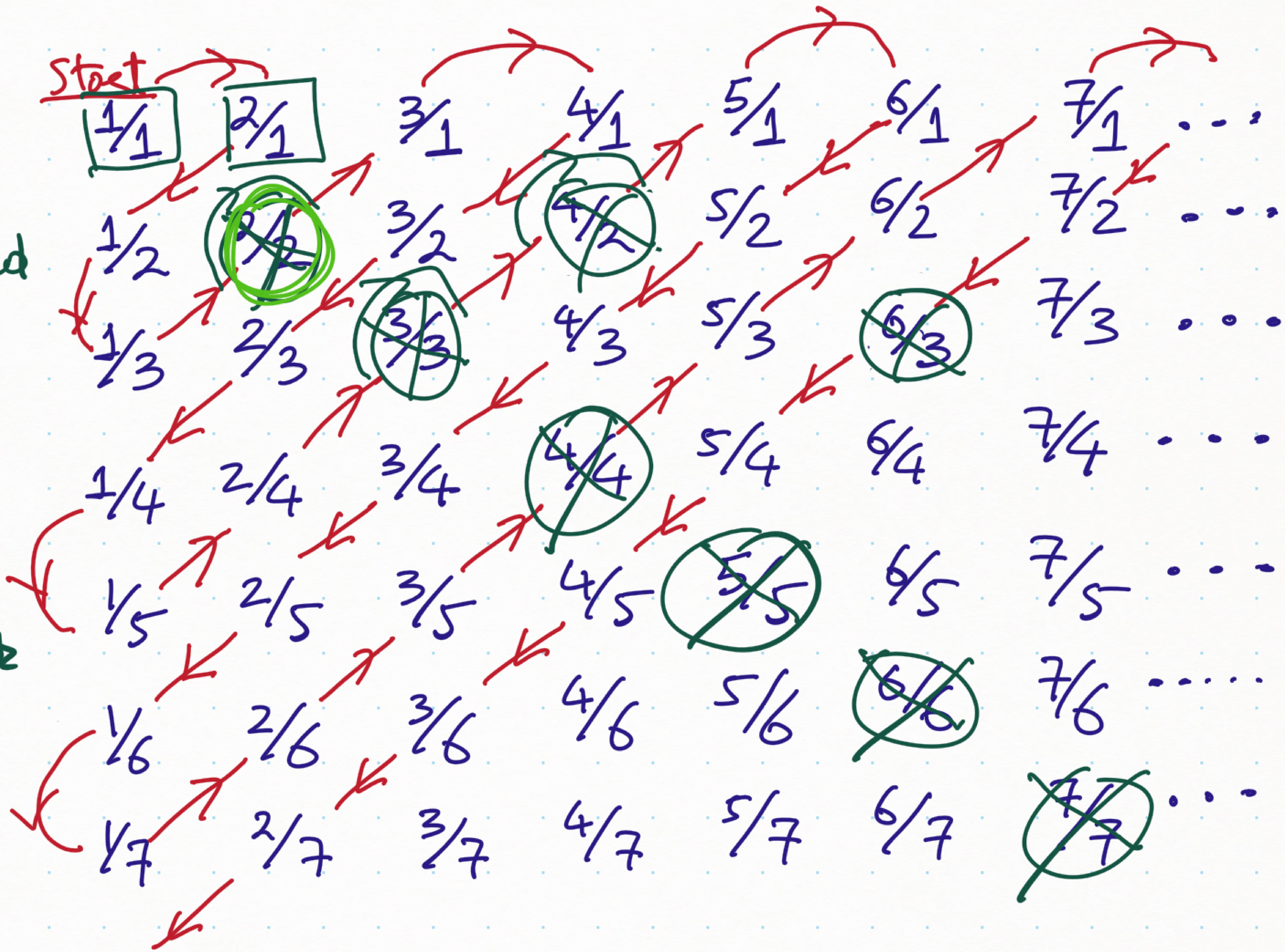
Start

1/1	2/1	3/1	4/1	5/1	6/1	7/1	...
1/2	2/2	3/2	4/2	5/2	6/2	7/2	...
1/3	2/3	3/3	4/3	5/3	6/3	7/3	...
1/4	2/4	3/4	4/4	5/4	6/4	7/4	...
1/5	2/5	3/5	4/5	5/5	6/5	7/5	...
1/6	2/6	3/6	4/6	5/6	6/6	7/6	...
1/7	2/7	3/7	4/7	5/7	6/7	7/7	...

Each rational is hit more than once: p/q is encountered in positions (p,q), $(2p,2q)$, $(3p,3q)$, ...

Easy Fix →

Just skip over a number already encountered



It is possible to even write an explicit formula for the bijection just shown!! (Try it!)

Here is another way:

set $A_1 = \{0\}$, $A_n = \{\pm p/q : p, q \in \mathbb{N}$ in the lowest terms with $p+q=n\}$

$A_1 = \{0\}$, $A_2 = \{\frac{1}{1}, \frac{-1}{1}\}$, $A_3 = \{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\}$,

$A_4 = \{\frac{1}{3}, \frac{-1}{3}, \frac{3}{1}, \frac{-3}{1}\}$, ...

Each A_n is finite & each rational appears in exactly one of these sets.

Bijection with $\mathbb{N}$ consecutively list the elements of A_1 followed by A_2 followed by A_3 . . . . .

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Real Analysis

Part #8

Theorem (Cantor 1874) $\mathbb{R}$ is uncountable

Proof [Cantor's Diagonalization Method (Cantor 1891)]

We prove something stronger

Interval $(0,1)$ is uncountable

(Think: Why is "$(0,1)$ uncountable" $\Uparrow \Updownarrow \Downarrow$ contrapositive "$\mathbb{R}$ uncountable" ?)

define a bijective function between $(0,1)$ & $\mathbb{R}$

Assume $(0,1)$ is countable

i.e, $\exists$ bijection $f : \mathbb{N} \to (0,1)$

$$
\begin{array}{lcll}
1 \longleftrightarrow f(1) & = & .a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15} \cdots \\
2 \longleftrightarrow f(2) & = & .a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{25} \cdots \\
3 \longleftrightarrow f(3) & = & .a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{35} \cdots \\
4 \longleftrightarrow f(4) & = & .a_{41}\ a_{42}\ a_{43}\ a_{44}\ a_{45} \cdots \\
5 \longleftrightarrow f(5) & = & .a_{51}\ a_{52}\ a_{53}\ a_{54}\ a_{55} \cdots \\
\vdots & \vdots & \vdots \quad \ddots
\end{array}
$$

list has all reals in $(0,1)$

Define $b \in (0,1)$ as $b = 0.b_1 b_2 b_3 b_4 \cdots$ as $b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$

claim b is not in the list above

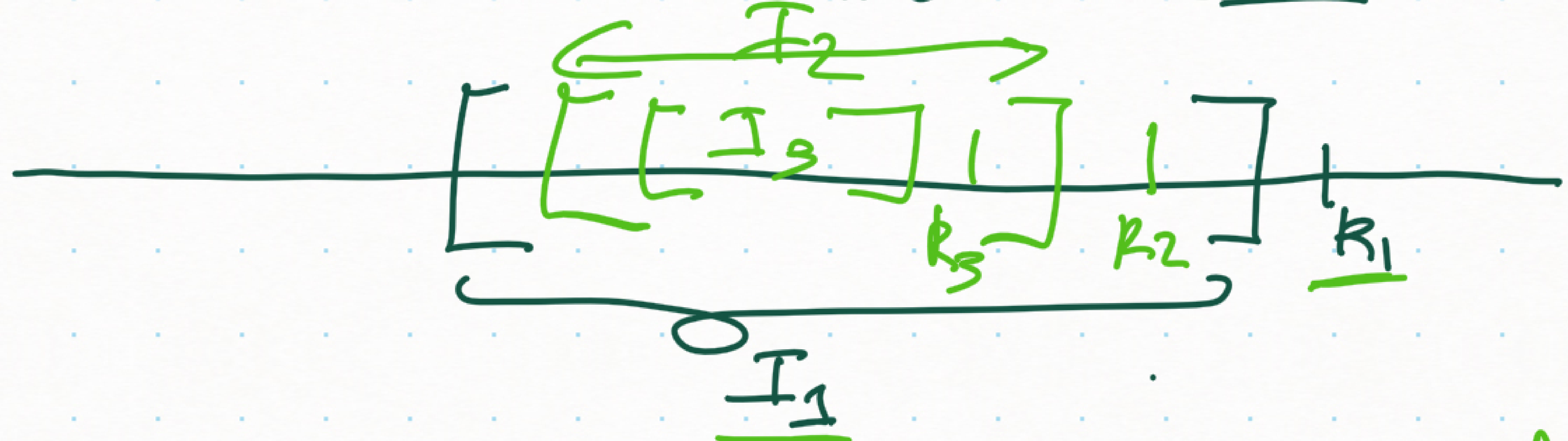
$\because b_i$ doesnot match with a_{ii} $\forall i$

Contradiction

[Proof of "$\mathbb{R}$ uncountable" by Cantor (1874)]

Assume $\mathbb{R}$ is countable, so it can be listed as
$\mathbb{R} = \{r_1, r_2, r_3, r_4, \ldots\}$

Let I_1 be a closed interval that does not contain r_1.



Let I_2 be a closed interval inside I_1 that does not contain r_2

⋮

Let I_{n+1} be a closed interval inside I_n that does not contain r_n

⋮

We defined $I_1, I_2, I_3, \cdots\cdots$ closed intervals

s.t. $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots \supseteq I_n \supseteq I_{n+1} \supseteq \cdots\cdots$

and $r_n \notin I_n \ \forall n$

Nested Interval Property tells us : $\bigcap_{n=1}^{\infty} I_n \neq \phi$

But, let r be any real number

Then r must equal some r_{n_0} in the list.

But $r \notin I_{n_0}$

which means $r \notin \bigcap_{n=1}^{\infty} I_n$

$\Rightarrow \bigcap_{n=1}^{\infty} I_n = \phi$ contrad.

Properties of countable sets (need proofs)

① If $A \subseteq B$ and B is countable then A is either countable or finite

② If $A_1, A_2, \ldots, A_m$ are each countable sets then $A_1 \cup A_2 \cup A_3 \cdots \cup A_m$ is countable.

③ If A_n is countable for each $n \in \mathbb{N}$ then $\bigcup_{n=1}^{\infty} A_n$ is countable.

"countable union of countable sets"

Cardinalities of Sets

We use the notation $|A|$ to denote the cardinality of set A.

We have shown $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$

we know $|\mathbb{N}| \neq |\mathbb{R}|$ (Cantor)

and $|\mathbb{N}| \leq |\mathbb{R}|$ ($\mathbb{N} \subseteq \mathbb{R}$)

Recall

① $|A| = |B| \iff \exists$ bijective function from A to B

② $|A| \leq |B| \iff \exists$ one-to-one function from A to B

③ $|A| \geq |B| \iff \exists$ onto function from A to B

Infinitely many Infinities

For a set A, the power set $\mathcal{P}(A)$ is the collection of all subsets of A.

e.g. $A = \{1, 2\}$
$\mathcal{P}(A) = \{\phi, \{1\}, \{2\}, A\}$
$f(1) = \{1\}, f(2) = \{2\}$

$$\mathcal{P}(A) = \{ B \mid B \subseteq A \}$$

(If $|A| = n$ then $|\mathcal{P}(A)| = 2^n$)

Ques Find a 1-1 mapping from A to $\mathcal{P}(A)$

$f: A \to \mathcal{P}(A)$, let $a \in A$, $f(a) = \{a\}$ check 1-1.

Ques What about an onto mapping from A to $\mathcal{P}(A)$

<u>Cantor's Thm</u> Let A be any set. Then
$$|A| < |\mathcal{P}(A)|$$

<u>Proof</u>

Proof by contradiction. Assume $|A| \geq |\mathcal{P}(A)|$

i.e. $\exists f: A \to \mathcal{P}(A)$ that is <u>onto</u>

Every subset of A appears as $f(a)$ for some $a \in A$.

$$B = \{a \in A : a \notin f(a)\}$$

$f(a)$ → subset of A

$f(a)$ may or may <u>not</u> include a

Since f is onto & $B \subseteq A$
then there must $\exists\, b \in A$ s.t. $f(b) = B$

Does $b \in B$? Then $b \notin f(b) = B$ (by defn of B)

Does $b \notin B$? Then $b \in f(b) = B$ (by defn of B)

contradictions

Corollary [$\exists$ infinitely many infinities]

$$|\mathbb{N}| < \underline{|\mathcal{P}(\mathbb{N})|} < \underline{|\mathcal{P}(\mathcal{P}(\mathbb{N}))|} < \underline{|\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))|} < \cdots$$

$\parallel$ $\longleftarrow$ Think!

$|\mathbb{R}|$