MATH 400

Real Analysis

Part #9

Defn A sequence is a function whose domain is $\mathbb{N}$

$$f : \mathbb{N} \to \mathbb{R}$$

$f(n)$ is the n^{th} term on the list

we often write $f(n)$ as a_n or x_n, etc.

Sometimes, a sequence is indexed to start from $n = n_0$

Most important definition

[Convergence of a sequence]

A sequence (a_n) converges to a real number a if for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_n - a| < \epsilon \ \forall n \geq N$.

↳ small positive real # ϵ

↳ term of seq. after which a_n is close to a

↳ distance between a_n and a is atmost ϵ

↳ for all n that are large enough

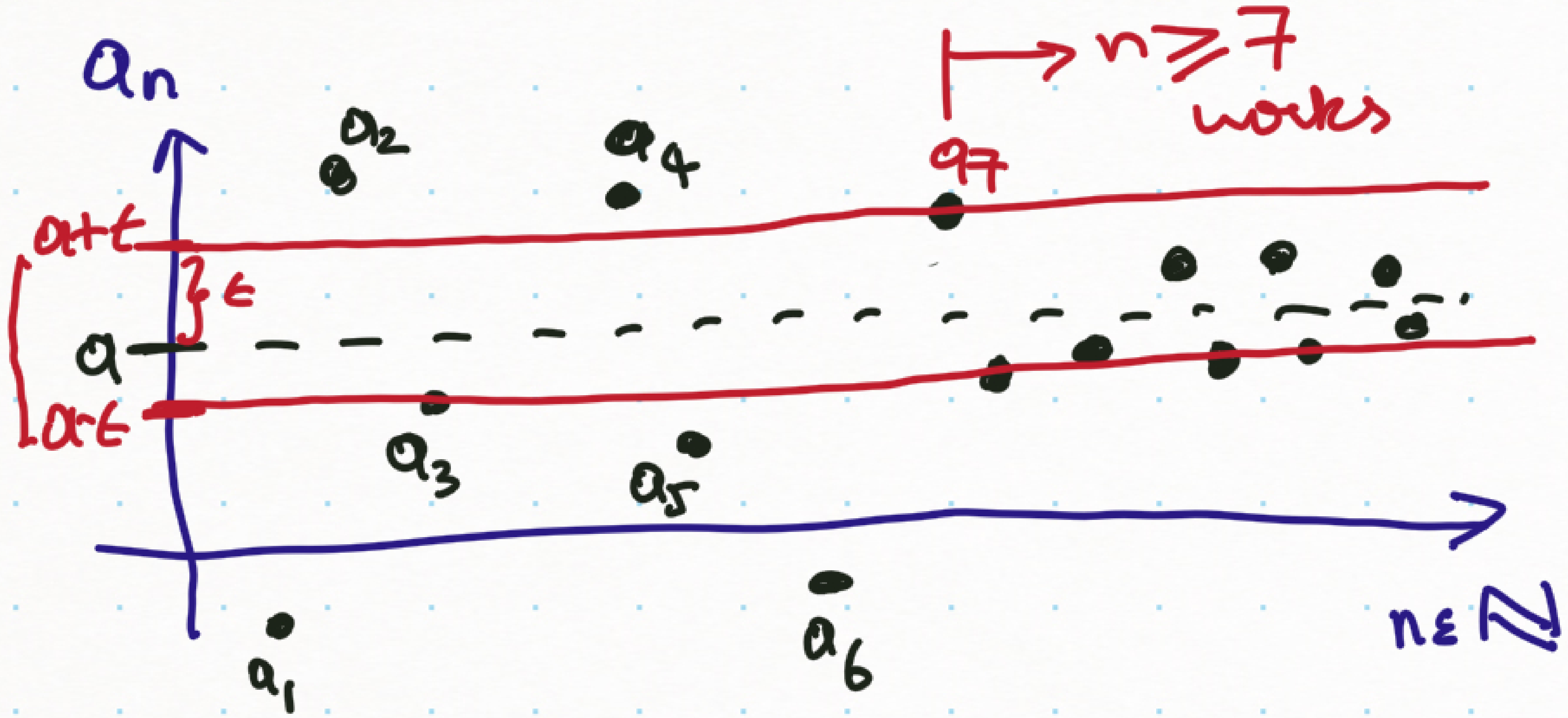
$a_n \longrightarrow a$ as $n \to \infty$ if $\forall \epsilon > 0, \exists N$ s.t. $|a_n - a| < \epsilon \ \forall n \geq N$

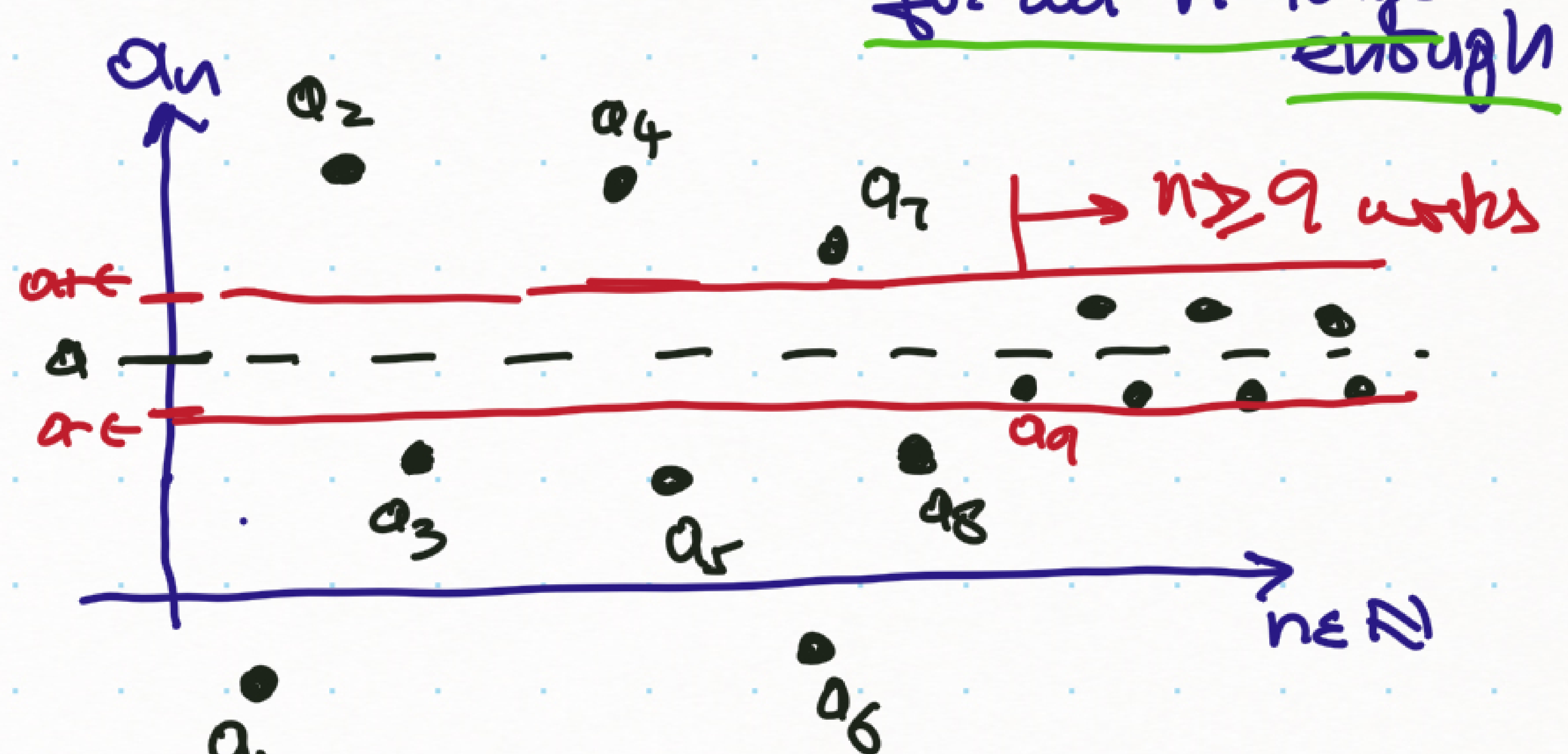
<u>Motivation</u> Recall we proved: $a = b \iff |a-b| < \epsilon \ \forall \epsilon > 0$

a equals b $\iff$ distance between a & b can be made arbitrarily small

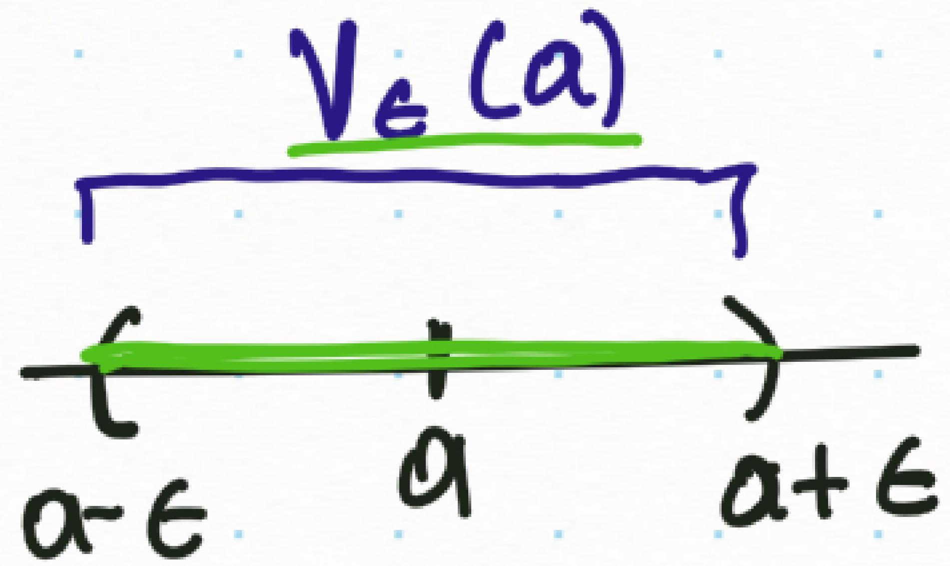
<u>Comment</u> $|a_n - a| < \epsilon \iff -\epsilon < a_n - a < \epsilon$

$\iff a - \epsilon < a_n < a + \epsilon \iff a_n \in (a-\epsilon, a+\epsilon)$ for all n large enough





Defn let $a \in \mathbb{R}$ and $\epsilon > 0$, the set $V_\epsilon(a) = \{x \in \mathbb{R} : |x-a| < \epsilon\}$ is called the <u>ϵ-neighborhood of a.</u>



Defn (<u>Convergence of a sequence: Topological version</u>)

A seq (a_n) converges to a if
given <u>any</u> <u>ϵ-neighborhood $V_\epsilon(a)$ of a</u>
there exists a point in the sequence
after which all of the terms are in $V_\epsilon(a)$.

<u>Remember</u> value of <u>N</u> depends on the choice of ϵ.

Template for proof $a_n \to a$

Step 0 Scratch work: Start with $|a_n - a| < \epsilon$
& unravel to solve for n
(in terms of ϵ)
This will tell us which N to choose.

Actual Proof

Step 1 Let $\epsilon > 0$

Step 2 Let $n > N =$ (where value for N comes from step 0)

Step 3 Redo the scratch work (without ϵ's)
but use the value of N to show
$n \geq N \Rightarrow |a_n - a| < \epsilon$

<u>example</u> Let $a_n = \frac{1}{n}$ for all n $(1, 1/2, 1/3, \dots\dots)$

Show $a_n \longrightarrow 0$ as $n \to \infty$.

[scratch work: we want $\boxed{|a_n - a| < \epsilon}$

i.e, $|\frac{1}{n} - 0| < \epsilon$

i.e., $\frac{1}{n} < \epsilon$, i.e. $\boxed{n > 1/\epsilon}$]

$\therefore$ choose $N = 1/\epsilon$

<u>solution</u> Let $\epsilon > 0$

Set $N \geq 1/\epsilon$

For any $n > N$,

$\rightarrow$ $|a_n - a| = |\frac{1}{n} - 0| = |\frac{1}{n}| = \frac{1}{n} < \frac{1}{N} \leq \frac{1}{1/\epsilon} = \epsilon$

since $n > N$

$\therefore |a_n - a| < \epsilon$.

<u>example</u> Let $a_n = \dfrac{3n+1}{n+2}$. Prove $\lim_{n\to\infty} a_n = 3$

[<u>scratchwork</u>: $|a_n - a| < \epsilon \Leftrightarrow \left|\dfrac{3n+1}{n+2} - 3\right| < \epsilon$

$\Leftrightarrow \left|\dfrac{3n+1}{n+2} - \dfrac{3(n+2)}{n+2}\right| < \epsilon \Leftrightarrow \left|\dfrac{3n+1-3n-6}{n+2}\right| < \epsilon$

$\Leftrightarrow \left|\dfrac{-5}{n+2}\right| < \epsilon \Leftrightarrow \dfrac{5}{n+2} < \epsilon \Leftrightarrow \dfrac{5}{\epsilon} < n+2$

$\Leftrightarrow \boxed{n > \dfrac{5}{\epsilon} - 2}$]

<u>Solution</u> Let $\epsilon > 0$

Set $N = \dfrac{5}{\epsilon} - 2$

[$n > N \Rightarrow n+2 > N+2$

$\Rightarrow \dfrac{1}{n+2} < \dfrac{1}{N+2} \Rightarrow \dfrac{5}{n+2} < \dfrac{5}{N+2}$]

For any $n > N$,

$|a_n - a| = \left|\dfrac{3n+1}{n+2} - 3\right| = \cdots = \left|\dfrac{-5}{n+2}\right| = \dfrac{5}{n+2} < \dfrac{5}{N+2}$

$\dfrac{5}{N+2} = \dfrac{5}{(\frac{5}{\epsilon} - 2) + 2} = \dfrac{5}{5/\epsilon} = \epsilon$

$\therefore |a_n - a| < \epsilon$.

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Real Analysis

Part #10

Theorem [Uniqueness of Limits]

The limit of a sequence, when it exists, is unique.

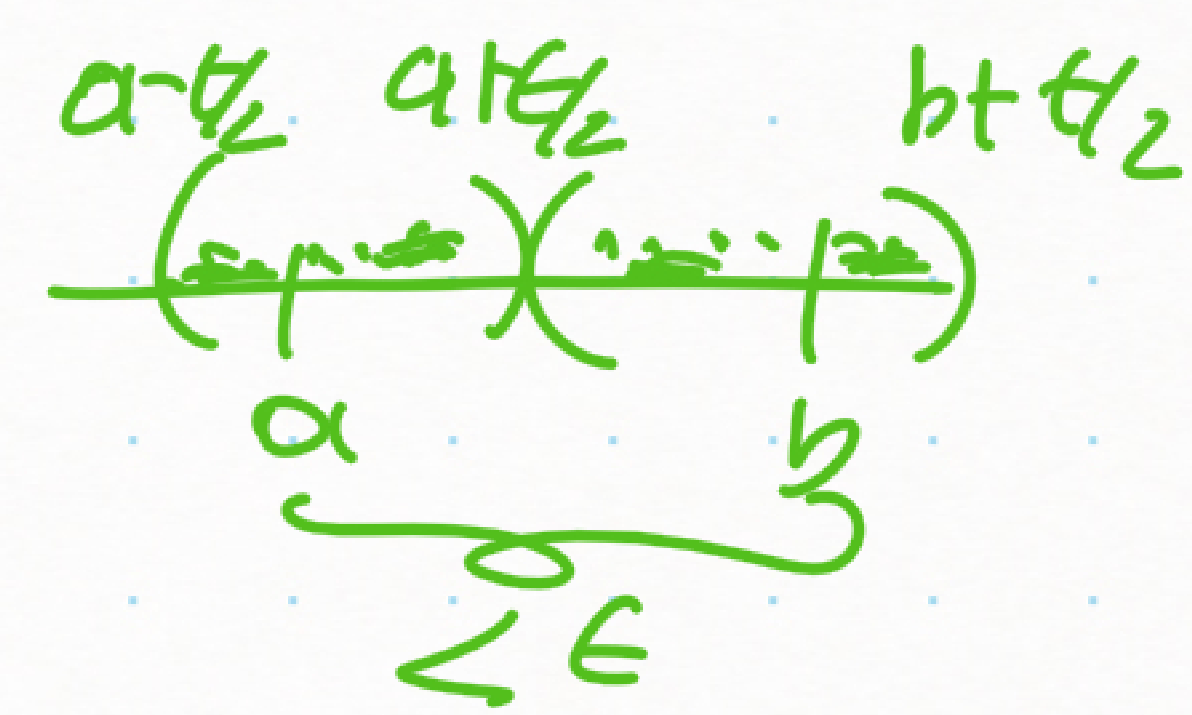
Proof Idea suppose $a_n \to a$ & $a_n \to b$

we want to show $a = b$

by $|a-b| < \epsilon \quad \forall \epsilon > 0$

How? as a_n gets closer to a it will be within distance $\epsilon/2$ of a

as a_n gets closer to b it will be within distance $\epsilon/2$ of b



Use Triangle inequality.

Theorem [Uniqueness of Limits]

The limit of a sequence, when it exists, is unique.

Proof Suppose $a_n \to a$ and $a_n \to b$

Let $\epsilon > 0$.

Since $\frac{\epsilon}{2} > 0$ and $a_n \to a$, $\exists N_1$ s.t. $|a_n - a| < \frac{\epsilon}{2}$ $\forall n > N_1$

Since $\frac{\epsilon}{2} > 0$ and $a_n \to b$, $\exists N_2$ s.t. $|a_n - b| < \frac{\epsilon}{2}$ $\forall n > N_2$

Let $n > N = ?$

$$|a-b| = |a - a_n + a_n - b| \leq ? = ? < ? = \epsilon$$

Fill in the blanks & discuss in class

Since $|a-b| < \epsilon$ for all $\epsilon > 0$, we have $a = b$ ∎

Defn A sequence that does not converge is said to diverge.

Three forms of divergence

- a_n diverges to ∞ if $\forall M > 0$, there exists N such that $a_n > M$ $\forall n > N$.

- a_n diverges to $-\infty$ if $\forall M < 0$, there exist N such that $a_n < M$ $\forall n > N$.

- Otherwise, (a_n)'s limit does not exist.

example $a_n = n^2$ show $\lim_{n\to\infty} a_n = \infty$

[scratch work: we want $a_n > M$

i.e, $n^2 > M$, i.e., $n > \sqrt{M}$]

solution Let $M > 0$.

Set $N = \sqrt{M}$

Then, for any $n > N$,

$$a_n = n^2 > N^2 = (\sqrt{M})^2 = M.$$

So, we have shown $a_n > M \ \forall n > N$.

<u>Comment</u> a_n diverges is same as $a_n \not\to a$ for any $a \in \mathbb{R}$.

What is the negation of definition of $a_n \to a$?

$\forall \epsilon > 0, \ \exists N$ s.t. $|a_n - a| < \epsilon \ \forall n > N$.

negation

$\exists \epsilon > 0, \ \forall N \ \exists n > N$ s.t. $|a_n - a| \geq \epsilon$.

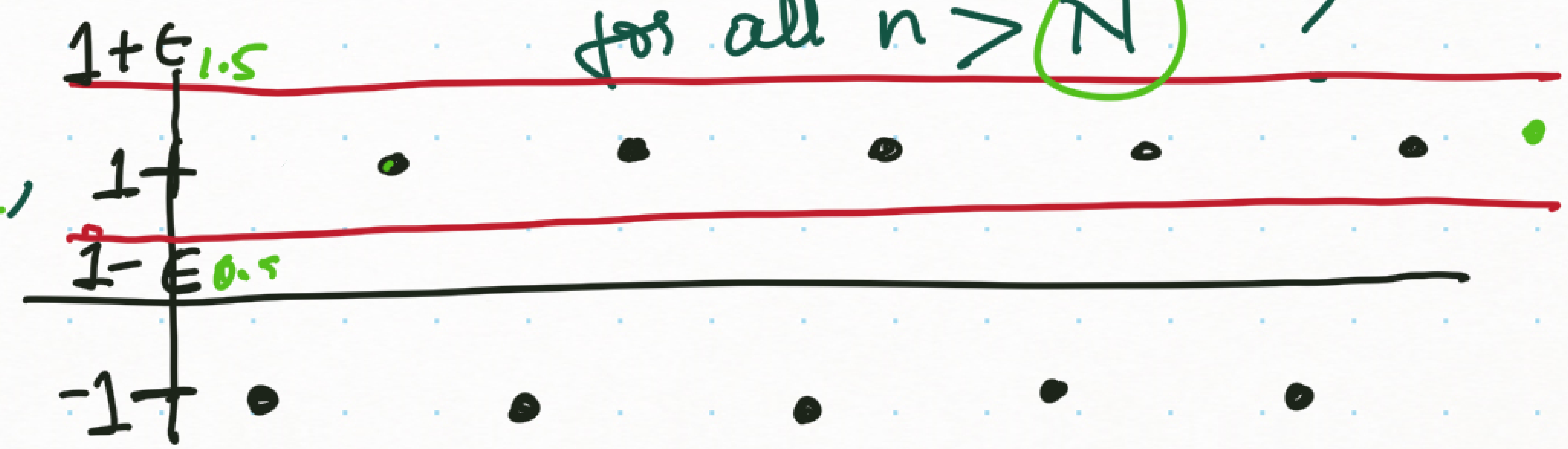
Find such a "bad" $\epsilon > 0$

example Let $a_n = (-1)^n$. Prove (a_n) diverges.

$(-1, 1, -1, 1, -1, 1, \ldots\ldots)$

[scratch work: what ϵ should we choose so that a_n is not within $(a-\epsilon, a+\epsilon)$ for all $n > N$?

Look at $a = 1$,



$\epsilon = \frac{1}{2}$ works

example Let $a_n = (-1)^n$. Prove (a_n) diverges.

Solution Suppose $a_n \to a$

Let $\epsilon = 1/2$

Since $a_n \to a$, there must exist N s.t. $|a_n - a| < \frac{1}{2}$ $\forall n > N$

i.e., $\boxed{|(-1)^n - a| < \frac{1}{2} \ \forall n > N.}$

case 1 n even: For $n > N$, we have

$|1-a| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < 1-a < \frac{1}{2} \Leftrightarrow -\frac{3}{2} < -a < -\frac{1}{2} \Leftrightarrow \frac{1}{2} < a < \frac{3}{2}$

case 2 n odd: For $n > N$, we have

$|-1-a| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < -1-a < \frac{1}{2} \Leftrightarrow \frac{1}{2} < -a < \frac{3}{2} \Leftrightarrow -\frac{3}{2} < a < -\frac{1}{2}$

we need $a \in (\frac{1}{2}, 3/2)$ & $a \in (-\frac{3}{2}, -\frac{1}{2})$ Not possible $\times$

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Real Analysis

Part #11

- Why study formal definitions?
- <u>Behavior of convergent sequences</u>

- Why study formal definitions?
- <u>Behavior of convergent sequences</u>

<u>Defn</u> A sequence (x_n) is <u>bounded</u> if
$\exists M > 0$ s.t. $|x_n| < M \;\; \forall n \in \mathbb{N}$

$-M < x_n < M$

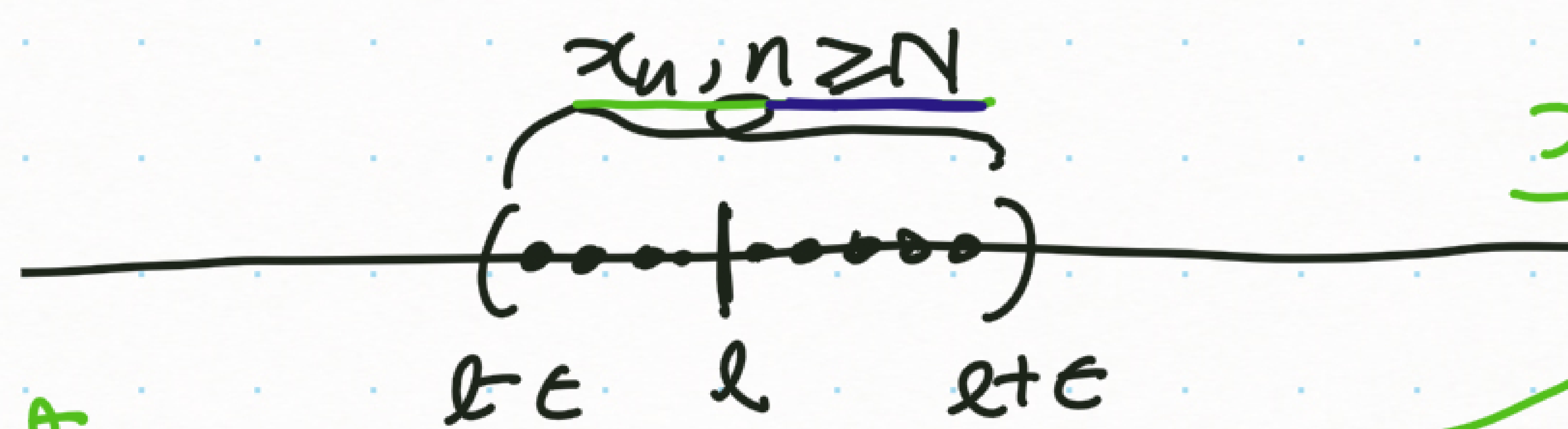
This means every $x_n \in [-M, M]$

<u>Theorem</u> Every convergent sequence is bounded.

what does the contrapositive tell us?

If (a_n) is unbdd. then (a_n) is divergent

$x_n \to \ell$



Theorem Every convergent sequence is bounded.

Proof Suppose $x_n \to l$

For every $\epsilon > 0$, say $\epsilon = 1$, $\exists\, N \in \mathbb{N}$ s.t.

$x_n \in (l-1, l+1)\ \forall n \geq N$

To avoid considering whether l is positive or negative, we can simply use the upper bound: $|x_n| < |l| + 1.\ \forall n \geq N.$

But, what about $x_1, x_2, \dots, x_{N-1}$?

$|x_N|, |x_{N+1}|, \dots$
$< |l| + 1$ ✓

Let $M = \max\{|x_1|, |x_2|, \dots, |x_{N-1}|, |l|+1\}$

$\Rightarrow |x_1| \leq M,\ |x_2| \leq M,\ \dots,\ |x_{N-1}| \leq M,\ |x_n| \leq M\ \forall n \geq N$

$\Rightarrow |x_n| \leq M\ \forall n \in \mathbb{N}.$

__Theorem [Algebra of limits]__ Let $\lim a_n = a$ & $\lim b_n = b$.

1. $\lim (ca_n) = ca$ for all $c \in \mathbb{R}$
2. $\lim (a_n + b_n) = a + b$
3. $\lim (a_n b_n) = ab$
4. $\lim (a_n / b_n) = a/b$, when $b \neq 0$.

__Proof ①__ Try it! Straightforward using $|ca_n - ca| = \underbrace{|c|}_{\text{fixed number}} \underbrace{|a_n - a|}_{\text{small since } a_n \to a}$

small $< \epsilon$

__Proof ②__ Try it! Straightforward using

$$\boxed{|(a_n + b_n) - (a+b)|} = |(a_n - a) + (b_n - b)| \quad \Delta \text{ ineq.}$$

$$\leq \underbrace{|a_n - a|}_{\text{small } \epsilon/2} + \underbrace{|b_n - b|}_{\text{small } \epsilon/2} < \epsilon$$

$< \epsilon$

Proof ④ $\left[\lim \frac{a_n}{b_n} = \frac{a}{b} \text{ if } b \neq 0\right]$

Look up details in the textbook.

If we can show that $\boxed{b_n \to b \Rightarrow \frac{1}{b_n} \to \frac{1}{b}}$ when $b \neq 0$

Then using ③ we have $a_n \frac{1}{b_n} \to a \frac{1}{b}$ $\frac{a_n}{b_n} \to \frac{a}{b}$

$$\left| \frac{1}{b_n} - \frac{1}{b} \right| = \left| \frac{b - b_n}{b\, b_n} \right| = \frac{|b - b_n|}{|b|\,|b_n|} \quad \text{want} < \epsilon$$

want $< \epsilon$ for $n \geq N$

$|b - b_n|$ small since $b_n \to b$

$|b|$ fixed numbers

why would $|b_n| \geq \delta > 0$?

$\forall n \geq N$

$|b_n| \to |b| \neq 0$

0   $|b|$

Ultimately, $|b_n|$ is going to be close to $|b| > 0$.

$\frac{1}{|b|\,|b_n|} <$ fixed numbes

i.e., $\frac{1}{|b_n|} <$ fixed numbes

i.e, $\boxed{|b_n| \geq \delta > 0}$ lower bd.

~~$|b_n| < M$~~

~~$\Rightarrow \frac{1}{|b_n|} > \frac{1}{M}$~~

~~we don't that~~

Proof of ③ [$a_n b_n \longrightarrow ab$]

Scratch work   We want to find N s.t. $|a_n b_n - ab| < \epsilon$
$\forall n \geq N$.

we know $a_n \to a$, i.e., $\forall \epsilon_1 > 0$, $\exists N_1$ s.t. $|a_n - a| < \epsilon_1$ $\forall n \geq N_1$

we know $b_n \to b$, i.e., $\forall \epsilon_2 > 0$, $\exists N_2$ s.t. $|b_n - b| < \epsilon_2$ $\forall n \geq N_2$

$|a_n b_n - ab| = |a_n b_n - ? + ? \quad - ab|$

Proof of ③ [$a_n b_n \to ab$]

Scratch work We want to find N s.T. $|a_n b_n - ab| < \epsilon$ $\forall n \geq N$.

we know $a_n \to a$, i.e., $\forall \epsilon_1 > 0$, $\exists N_1$ s.t. $|a_n - a| < \epsilon_1$ $\forall n \geq N_1$

we know $b_n \to b$, i.e., $\forall \epsilon_2 > 0$, $\exists N_2$ s.t. $|b_n - b| < \epsilon_2$ $\forall n \geq N_2$

$$\underbrace{|a_n b_n - ab|}_{\text{small } <\epsilon?} = |a_n b_n - ab_n + ab_n - ab|$$

$$\leq |a_n b_n - ab_n| + |ab_n - ab| \qquad \triangle \text{ ineq.}$$

$$= |a_n - a|\,|b_n| + |a|\,|b_n - b|$$

we can make this small since $a_n \to a$

$|a_n - a| < \epsilon_1 = \frac{\epsilon}{2C}$

b_n is bdd. $|b_n| \leq C$

$\frac{\epsilon}{2C} C = \epsilon/2$

fixed

$b_n \to b$, we can make this small

$|b_n - b| < \epsilon_2 = \frac{\epsilon}{2|a|}$

$|a| \frac{\epsilon}{2|a|} = \epsilon/2$

Proof of ③. Let $\epsilon > 0$

Since (b_n) converges, we know (b_n) is bounded
i.e., $\exists C$ s.t. $|b_n| \leq C \; \forall n$.

Let $\epsilon_1 = \frac{\epsilon}{2C+1}$. Since $\epsilon_1 > 0$, $\exists N_1$ s.t. $\underline{|a_n - a| < \epsilon_1 \; \forall n \geq N_1}$

Let $\epsilon_2 = \frac{\epsilon}{2|a|+1}$. Since $\epsilon_2 > 0$, $\exists N_2$ s.t. $\underline{|b_n - b| < \epsilon_2 \; \forall n \geq N_2}$

Let $n > N = \max\{N_1, N_2\}$; then

$$
\begin{aligned}
|a_n b_n - ab| &= |a_n b_n - ab_n + ab_n - ab| \\
&\leq |a_n b_n - ab_n| + |ab_n - ab| \\
&= \underline{|a_n - a|\,|b_n|} + \underline{|a|\,|b_n - b|} \\
&< \epsilon_1 C + |a| \epsilon_2 = \frac{\epsilon}{2C+1} C + |a| \frac{\epsilon}{2|a|+1} \\
&< \epsilon/2 + \epsilon/2 = \epsilon
\end{aligned}
$$

$n \geq N_1$ and $n \geq N_2$

■

example $\lim_{n\to\infty} \frac{1}{2}\left(\frac{1/n + 1/n^2 + 4}{5 - 1/n^2}\right)\left(\frac{3n+1}{n+2} + \frac{1}{\sqrt{n}}\right) = ??$

we know $\left(1/n\right) \to 0$, $\left(1/n^2\right) \to 0$,

$\left(5 - 1/n^2\right) \to 0$, $\left(\frac{3n+1}{n+2}\right) \to 3$, $\left(\frac{1}{\sqrt{n}}\right) \to 0$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{2}\left(\frac{1/n + 1/n^2 + 4}{5 - 1/n^2}\right)\left(\frac{3n+1}{n+2} + \frac{1}{\sqrt{n}}\right) = \frac{1}{2}\left(\frac{0+0+4}{5}\right)(3+0)$$

$$= \frac{6}{5}.$$

Theorem [Order Limit Theorem] Let $a_n \to a$ & $b_n \to b$.

① If $a_n \geq 0$ $\forall n \in \mathbb{N}$ then $a \geq 0$

② If $a_n \leq b_n$ $\forall n \in \mathbb{N}$ then $a \leq b$

③ If $\exists c \in \mathbb{R}$ s.t. $c \leq b_n$ $\forall n$, then $c \leq b$.
If $\exists d \in \mathbb{R}$ s.t. $a_n \leq d$ $\forall n$, then $a \leq d$.

Theorem [Order Limit Theorem] Let $a_n \to a$ & $b_n \to b$.

① If $a_n \geq 0$ $\forall n \in \mathbb{N}$ then $a \geq 0$

② If $a_n \leq b_n$ $\forall n \in \mathbb{N}$ then $a \leq b$

③ If $\exists c \in \mathbb{R}$ s.t. $c \leq b_n$ $\forall n$, then $c \leq b$.
If $\exists d \in \mathbb{R}$ s.t. $a_n \leq d$ $\forall n$, then $a \leq C$.

Proof ① Think about it!

② $(b_n - a_n) \longrightarrow b - a$ (by algebra of limits)
& since $b_n - a_n \geq 0$, part ① $\Rightarrow$ $b - a \geq 0$, i.e., $b \geq a$.

③ Let $a_n = c$ $\forall n$ & apply part ②.
Let $b_n = d$ $\forall n$ & apply part ②.