MATH 400

Real Analysis

Part #12

We know: Convergent sequences are bounded.
But not all bounded seq.s are convergent.
(Recall examples)

Ques Under what condition is a bdd. seq. convergent.

Defn A sequence (a_n) is **increasing** if $a_n \leq a_{n+1}$ $\forall n$
and **decreasing** if $a_n \geq a_{n+1}$ $\forall n$
A sequence is **monotone** if it is either increasing or decreasing.

Monotone Convergence Theorem
If a sequence is monotone and bounded, then it converges

MCT If a sequence is monotone and bounded then it converges.

Proof Let (a_n) be montone and bounded.

Let us assume (a_n) is increasing (proof for (a_n) decreasing is similar)

Consider the set of points $\{a_n : n \in \mathbb{N}\}$

By assumption, this set is bounded ($\because$ (a_n) is bounded)

Where do you think (a_n) converges to?

MCT If a sequence is monotone and bounded then it converges.

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Consider the set of points $\{a_n : n \in \mathbb{N}\}$

By assumption, this set is bounded ($\because (a_n)$ is bounded)

$$\text{Let } s = \sup\{a_n : n \in \mathbb{N}\}$$

Claim $\lim_{n\to\infty} a_n = s$

Proof Let $\epsilon > 0$. Since s is the least u.b., $s-\epsilon$ is not an u.b.

$\therefore \exists\, a_N$ s.t. $s-\epsilon < a_N$

Since (a_n) is increasing, $a_N \leq a_n \ \forall n \geq N$. $\therefore s-\epsilon < a_N \leq a_n \leq s < s+\epsilon$

i.e. $|a_n - s| < \epsilon \ \forall n \geq N$

What are the different behaviors of monotone sequences?

e.g. $(n) = (1, 2, 3, 4, \ldots)$ is an inc. seq. that diverges to ∞

$(\frac{1}{n}) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots)$ is a dec. seq. that converges to 0

Corollary (to MCT) Suppose (a_n) is monotone. Then

(a_n) converges $\iff$ (a_n) is bounded

Moreover,

- If (a_n) is increasing, then either (a_n) diverges to ∞ or $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$
- If (a_n) is decreasing, then either (a_n) diverges to $-\infty$ or $\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$

example Let (a_n) be the sequence defined as

$a_1 = 0.1$, $a_2 = 0.12$, $a_3 = 0.123$, $a_4 = 0.1234$, ... ,

$a_{11} = 0.1234567891011$, $a_{12} = 0.123456789101112$, ...

Prove that (a_n) converges.

Proof a_{n+1} and a_n match exactly until the last digits of a_{n+1}, which are the digits of $n+1$.

$a_{n+1} - a_n$ is number $0.00\ldots0$ digits of $n+1$ > 0

i.e, $a_{n+1} - a_n > 0 \Leftrightarrow a_{n+1} > a_n \; \forall n$ $\therefore (a_n)$ inc. seq.

$0.1 \leq a_n \leq 0.2 \; \forall n$ $\therefore (a_n)$ is bdd.

By MCT (a_n) is convergent.

Math 400

Real Analysis

Part #13

A Detour to Series

Defn Let (b_n) be a sequence.
An infinite series is a formal expression

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \cdots\cdots$$

The corresponding sequence of partial sums (s_m) is defined to be $s_m = b_1 + b_2 + \cdots + b_m$, $\forall m$

We say $\sum_{n=1}^{\infty} b_n$ converges to a number B

if (s_m) converges to B

example Consider $\sum_{n=1}^{\infty} 1/n^2$

The seq. of partial sums (S_m) is $S_m = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2}$

Since each term of the series is positive, (S_m) is increasing.

$S_1 = 1$, $S_2 = 1 + \frac{1}{4}$, $S_3 = 1 + \frac{1}{4} + \frac{1}{9}$, ...

Is (S_m) bounded?

$$S_m = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \cdots + \frac{1}{m\cdot m}$$

$$< M \quad \forall m$$

example Consider $\sum_{n=1}^{\infty} 1/n^2$ is convergent

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Is (s_m) bounded?

$$s_m = 1 + \frac{1}{2\cdot 2} + \frac{1}{3\cdot 3} + \frac{1}{4\cdot 4} + \cdots + \frac{1}{m\cdot m}$$

$$< 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3} + \cdots + \frac{1}{m\cdot (m-1)}$$

$$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{(m-1)} - \frac{1}{m}\right)$$

$$= 1 + 1 - \frac{1}{m} = 2 - 1/m$$

$$< 2 \quad \forall m.$$

$\therefore$ By MCT (s_m) converges.

*Example (Harmonic Series) $\sum_{n=1}^{\infty} \frac{1}{n}$

Since each term is positive, seq. of partial sums (S_m)
$$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}$$ is increasing.

Is (S_m) bounded?

Is 2 an upper bound?

Is 2.5 an upper bound?

Can any fixed number be an upper bound?

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Is (S_m) bounded?

Is 2 an upper bound? $S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$

Is 2.5 an upper bound? $S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \cdots + \frac{1}{8}\right)$

$$= 2\tfrac{1}{2}.$$

Can any fixed number be an upper bound?

*Example (Harmonic Series) $\sum_{n=1}^{\infty} \frac{1}{n}$

Since each term is positive, seq. of partial sums (S_m)

$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}$ is increasing.

Is (S_m) bounded?

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Is 2.5 an upper bound? $S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) = 2\frac{1}{2}$ ✓

Can any fixed number be an upper bound?

$$S_{2^k} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^{k-1}+1} + \dots + \frac{1}{2^k}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) + \dots + \left(\frac{1}{2^k} + \dots + \frac{1}{2^k}\right)$$

$$= 1 + \frac{1}{2} + 2\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \dots + 2^{k-1}\left(\frac{1}{2^k}\right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} = 1 + k\left(\frac{1}{2}\right)$$

Is 24 an upper bd.?

Pick $k = 48$,

$S_{2^{48}} > 1 + 48\left(\frac{1}{2}\right) = 25$

So, ~~48~~ 24 cannot be an u.bound for (S_m).

This argument using $S_{2^k} > 1 + k\left(\frac{1}{2}\right)$ show (S_m) is unbdd.

$\Rightarrow (S_m)$ is not convergent ($\because$ convergent $\Rightarrow$ bdd.)

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

The Harmonic Series argument is a special instance of the following:

Theorem [Cauchy Condensation Test]

Suppose (b_n) is decreasing and $b_n \geq 0 \ \forall n$. Then,

$$\sum_{n=1}^{\infty} b_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n b_{2^n} \text{ converges}$$

Proof $\boxed{\Rightarrow}$ Prove the contrapositive as we did with Harmonic Series

$$\sum_{n=0}^{\infty} 2^n b_{2^n} \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ diverges}$$

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$$\sum_{n=1}^{\infty} b_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n b_{2^n} \text{ converges}$$

Proof $[\Leftarrow]$ Assume $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges

Then the seq. of partial sums (t_k), $t_k = b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}$ converges, and hence is bounded: $\exists M > 0$ s.t. $t_k \leq M \ \forall k$.

To show $\sum_{n=1}^{\infty} b_n$ converges, we only need to show $s_m = b_1 + b_2 + \cdots + b_m$ is a bounded seq. ($\because$ we already know (s_m) is increasing).

Fix m & pick k s.t. $m \leq 2^{k+1} - 1$. Then $s_m \leq s_{2^{k+1}-1}$

$$s_{2^{k+1}-1} = b_1 + (b_2 + b_3) + (b_4 + \cdots + b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1}) \leq ?$$

$$s_{2^{k+1}-1} = b_1 + (b_2+b_3) + (b_4+b_5+b_6+b_7) + \cdots + (b_{2^k} + \cdots + b_{2^{k+1}-1})$$

$$\leq b_1 + (b_2+b_2) + (b_4 + \cdots + b_4) + \cdots + (b_{2^k} + \cdots + b_{2^k})$$

(Why?)

since (b_n) dec.

$b_1 \geq b_2 \geq b_3 \geq \cdots$

$$= b_1 + 2b_2 + 4b_4 + \cdots + 2^k b_{2^k}$$

$$= t_k$$

Since $s_m \leq s_{2^{k+1}-1} \leq t_k \leq M \Rightarrow s_m \leq M \ \forall m$

we get (s_m) is bounded.

$\therefore$ by MCT, (s_m) converges. So, $\sum_{n=1}^{\infty} b_n$ converges.

Cor $\sum_{n=1}^{\infty} 1/n^p$ converges if and only if $p > 1$.

Proof Think!

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Part #14

We showed seq. of partial sums (s_m) of the Harmonic Series does not converge by analyzing one particular sequence within (s_m)

namely, $(S_{2^k}) = (s_1, s_2, s_4, s_8, s_{16}, \ldots\ldots)$ ← subsequence

We showed seq. of partial sums (s_m) of the Harmonic Series does not converge by analyzing one particular sequence within (s_m) namely, $(S_{2^k}) = (s_1, s_2, s_4, s_8, s_{16}, \ldots\ldots)$ ↑ subsequence

Defn Let (a_n) be a sequence of real numbers. Let $n_1 < n_2 < n_3 < \cdots < \ldots$ be an increasing sequence of natural numbers.

Then the sequence $(a_{n_1}, a_{n_2}, a_{n_3}, \ldots\ldots)$ is called a subsequence of (a_n), and is denoted by (a_{n_k}), $k \in \mathbb{N}$.

$(a_1, a_2, a_3, \ldots\ldots)$

$n_1 < n_2 < n_3 < n_4 < \ldots$

$(a_{n_1}, a_{n_2}, a_{n_3}, \ldots\ldots)$

The order of appearance does not change.

And no terms are repeated.

e.g $(a_n) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots\right)$

which of these is a subsequence of (a_n)?

① $\left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots\right)$ Yes

② $\left(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \ldots\right)$ Yes

③ $\left(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{50}, \frac{1}{1000}, \ldots\right)$ No

$\frac{1}{5}$ before $\frac{1}{10}$ $\quad$ $\frac{1}{50}$ before $\frac{1}{100}$

④ $\left(1, \frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots\right)$ No

Theorem If a sequence (a_n) converges to a then every subsequence of (a_n) also converges to a.

Proof

Let (a_{n_k}) be any subseq. of (a_n).

Given $\epsilon > 0$, $\exists\, N$ s.t. $|a_n - a| < \epsilon \;\; \forall n \geq N$. (Since $(a_n) \to a$)

Since $1 \leq n_1 < n_2 < n_3 < \cdots$, is an inc. seq. of natural numbers, $k \leq n_k$ for all k.

$\therefore\; |a_{n_k} - a| < \epsilon \;\; \forall k \geq N$ $\quad (\because k \geq N \Rightarrow n_k \geq N \Rightarrow |a_{n_k} - a| < \epsilon)$

since $n_k \geq k \geq N$

i.e., $(a_{n_k}) \to a$ as $k \to \infty$.

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Let (a_{n_k}) be any subseq. of (a_n).

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i.e., $(a_{n_k}) \to a$ as $k \to \infty$.

Cor Sequence (a_n) converges to a $\Leftrightarrow$ every subsequence of (a_n) converges to a

Divergence Criterion If a sequence contains two subsequences that converge to different limits, then the sequence cannot be convergent

e.g. Consider the sequence. $\left(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, \dots\right)$

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$\left(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \ldots\right)$ is a subsequence that converges to $1/5$

$\left(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \ldots\right)$ is a subsequence that converges to $-1/5$

$\therefore$ the original sequence is not convergent

<u>example</u> What is the limit of (b^n) where $0<b<1$?

example What is the limit of (b^n) where $0<b<1$?

$$b^1 > b^2 > b^3 > \dots > 0$$

$\Rightarrow$ (b^n) is decreasing & bounded below.

By MCT, $(b^n) \to l$ for some $l \in [0, b] \subseteq [0, 1)$

How to find l?

example What is the limit of (b^n) where $0<b<1$?

$$b^1 > b^2 > b^3 > \dots > 0$$

$\Rightarrow$ (b^n) is decreasing & bounded below.

By MCT, $(b^n) \to l$ for some $l \in [0, b] \subseteq [0, 1)$

How to find l?

(b^{2n}) is a subsequence. $\therefore$ $(b^{2n}) \to l$ ① (By Previous Thm.)

Also, $b^{2n} = b^n b^n$, product of two seq.'s. $\therefore$ $(b^{2n}) \to l \cdot l = l^2$ ②

(By Algebra of limits)

$\therefore$ ① = ② gives $l^2 = l \Leftrightarrow l^2 - l = 0 \Leftrightarrow l(l-1) = 0$ $\Rightarrow$ $l = 0$.

by uniqueness of limits

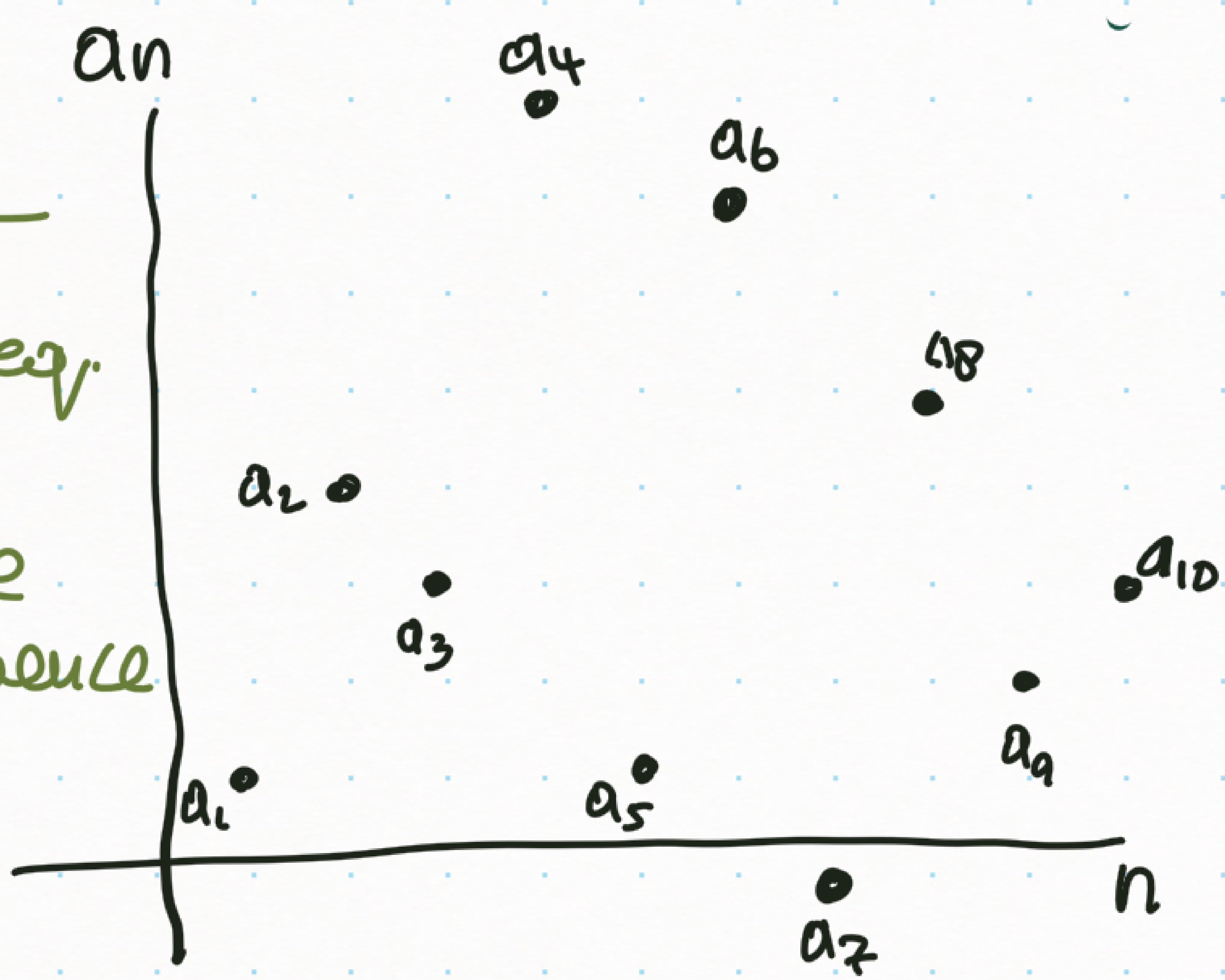
Can we always find a convergent subsequence even if the sequence is not convergent?

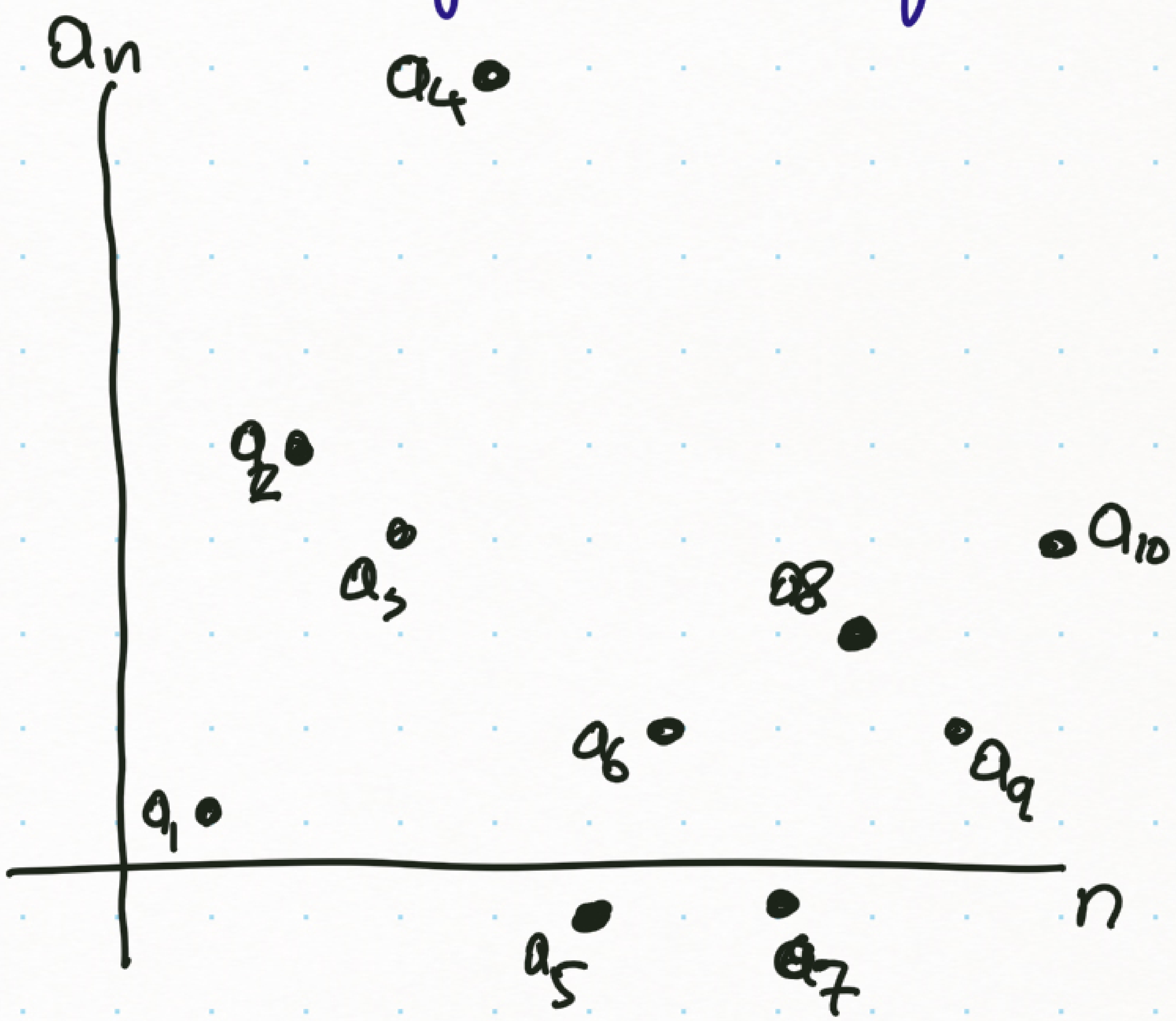
Bolzano – Weierstrass Theorem

Every **bounded** sequence contains a convergent subsequence.

Idea

Every seq. has a monotone subsequence





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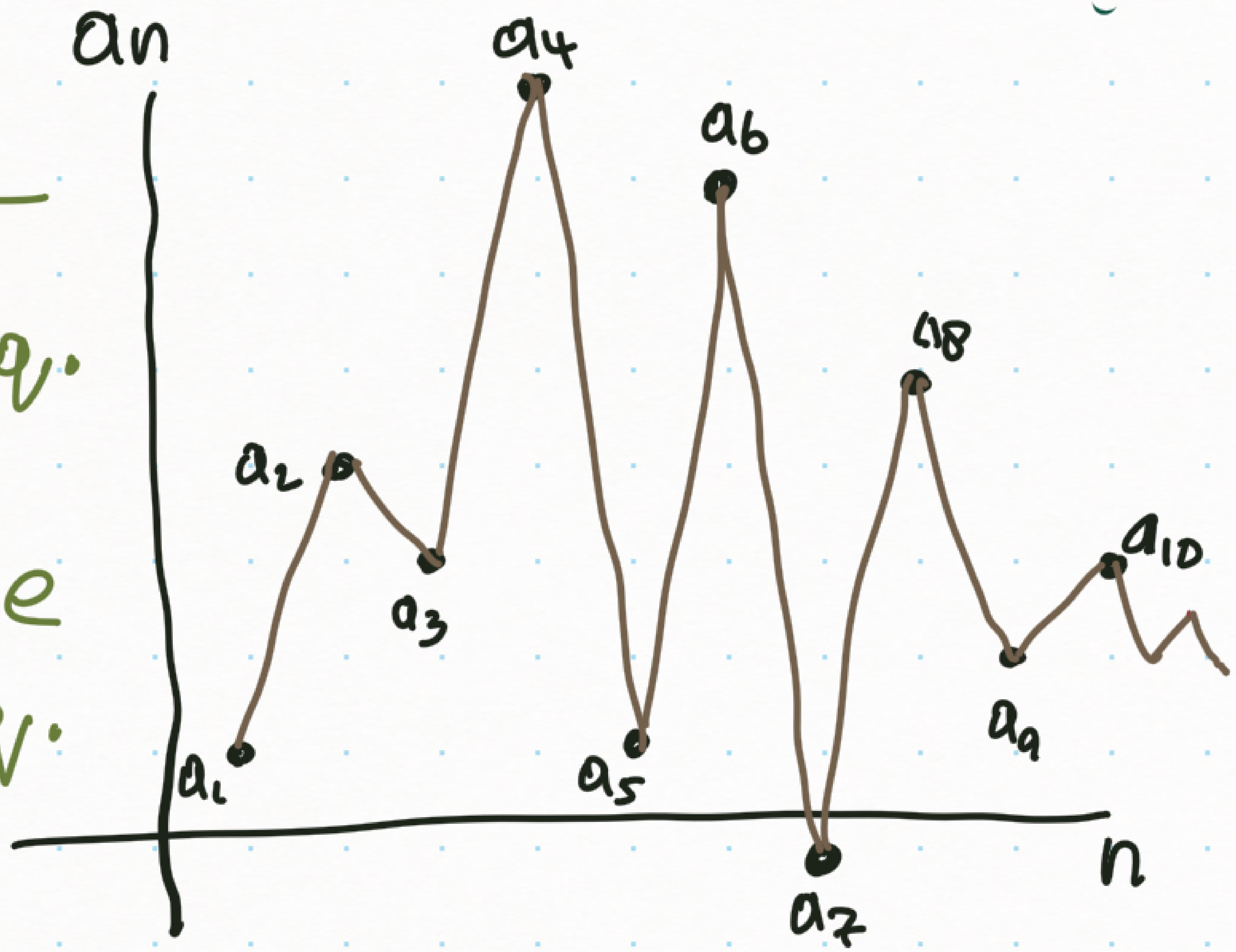

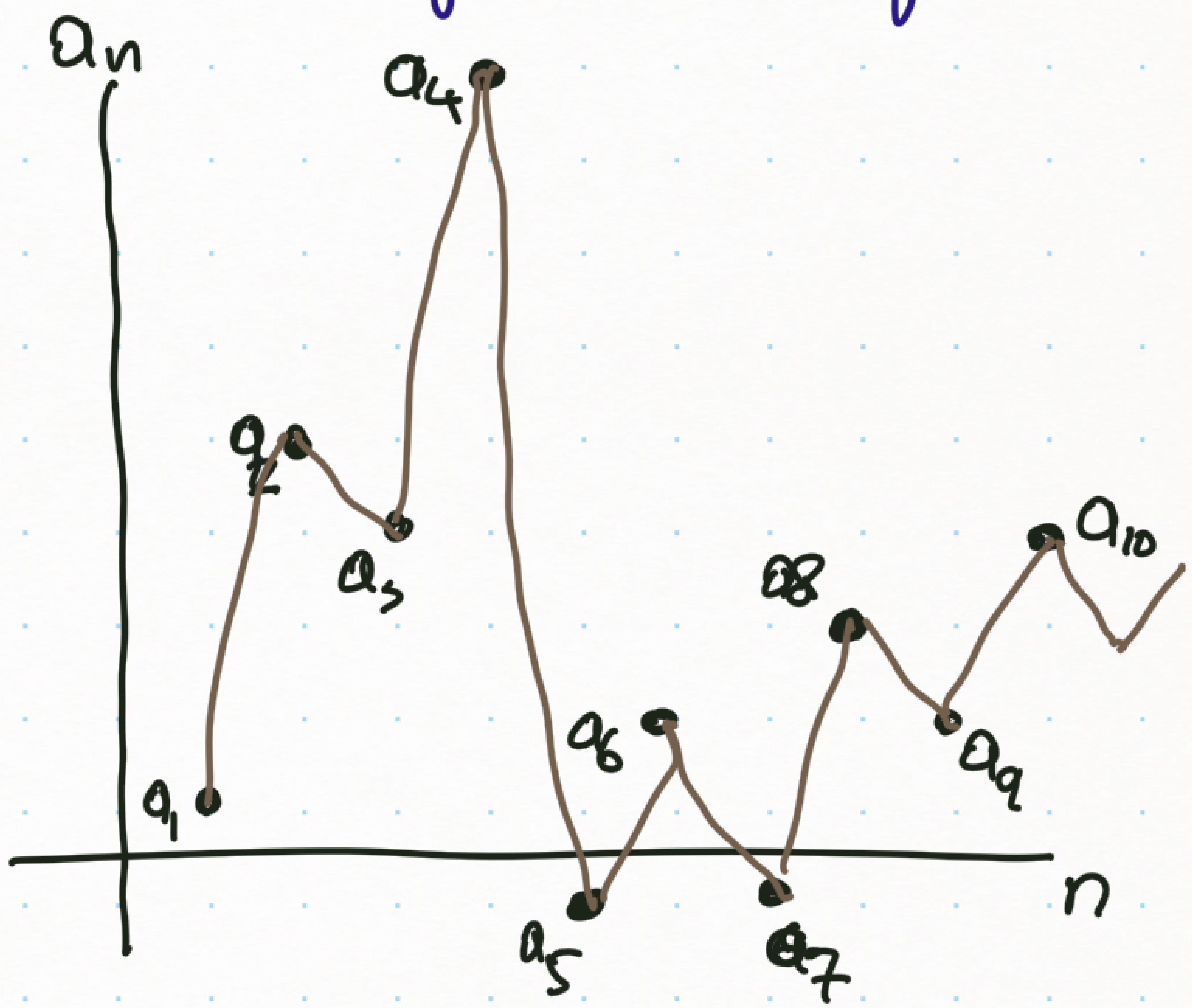

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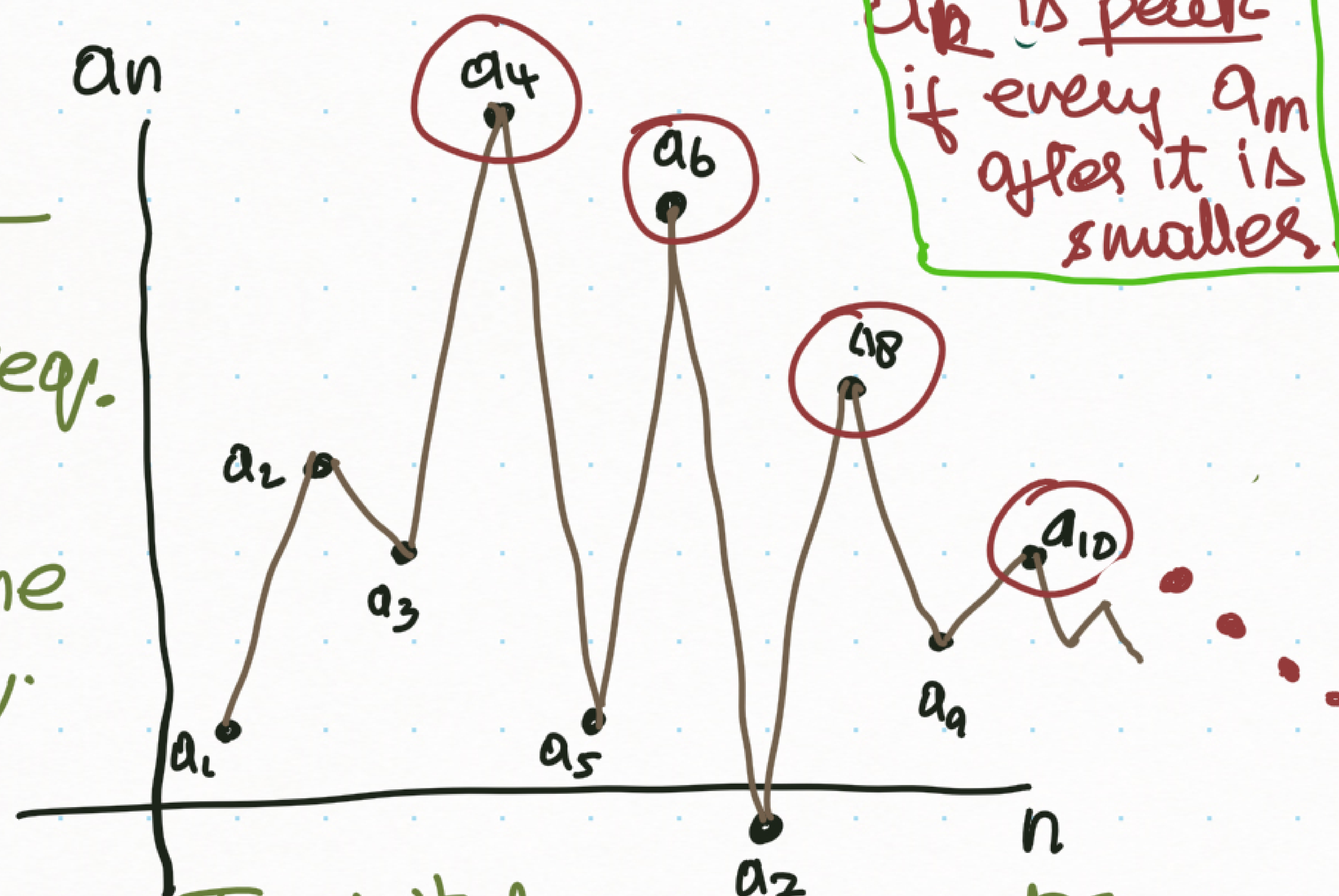
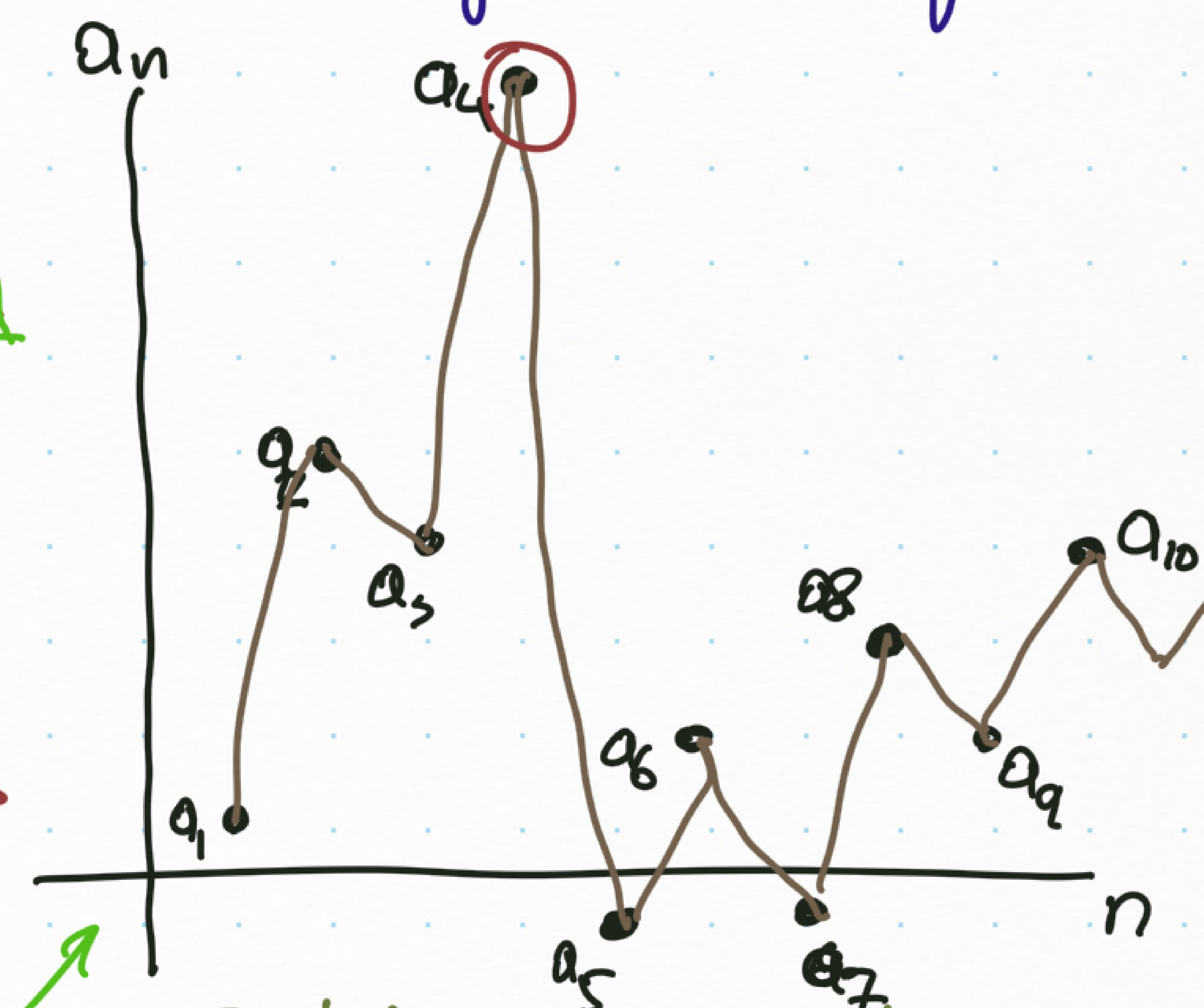
a_k is peak if every a_m after it is smaller.



Infinitely many peaks



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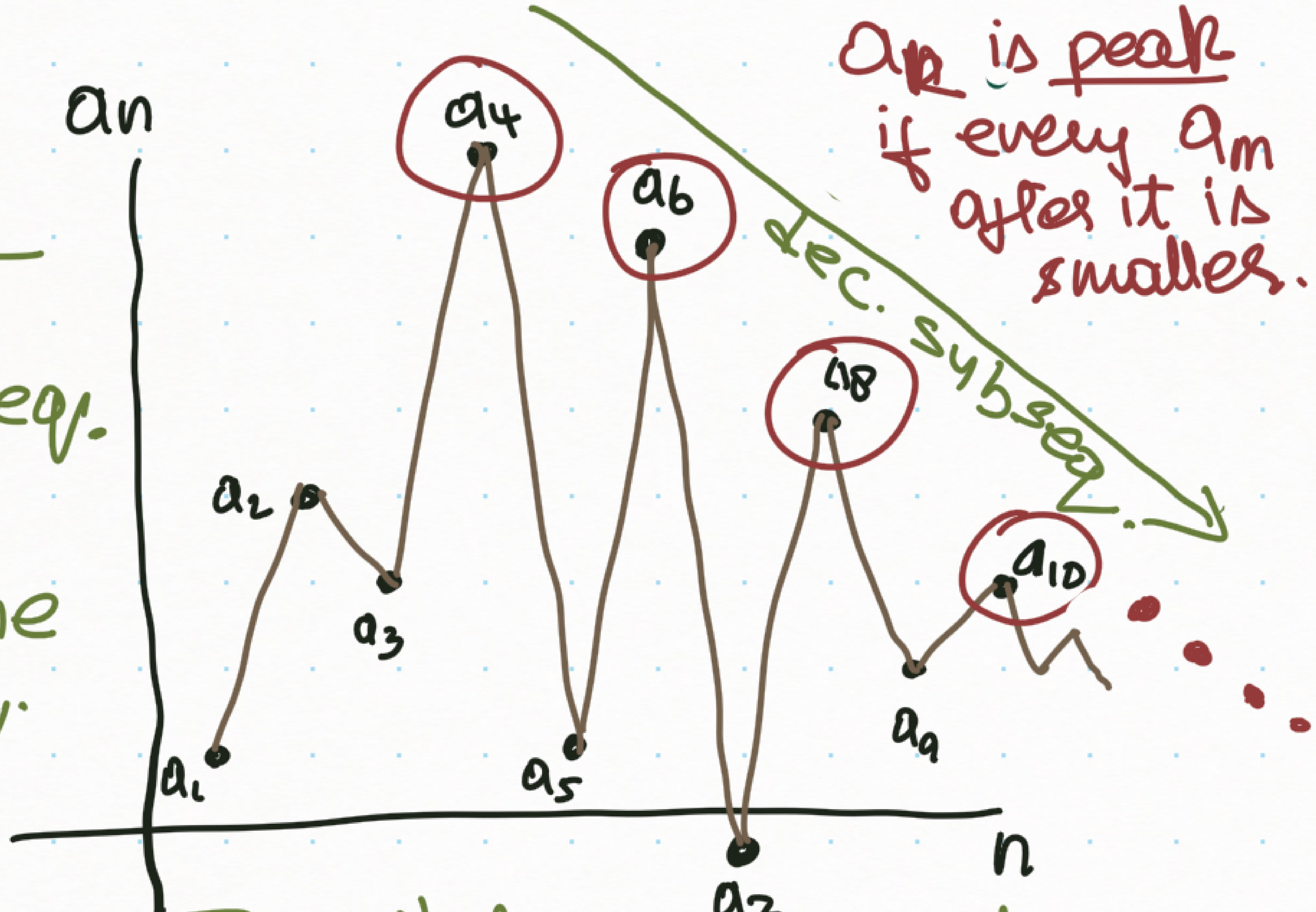

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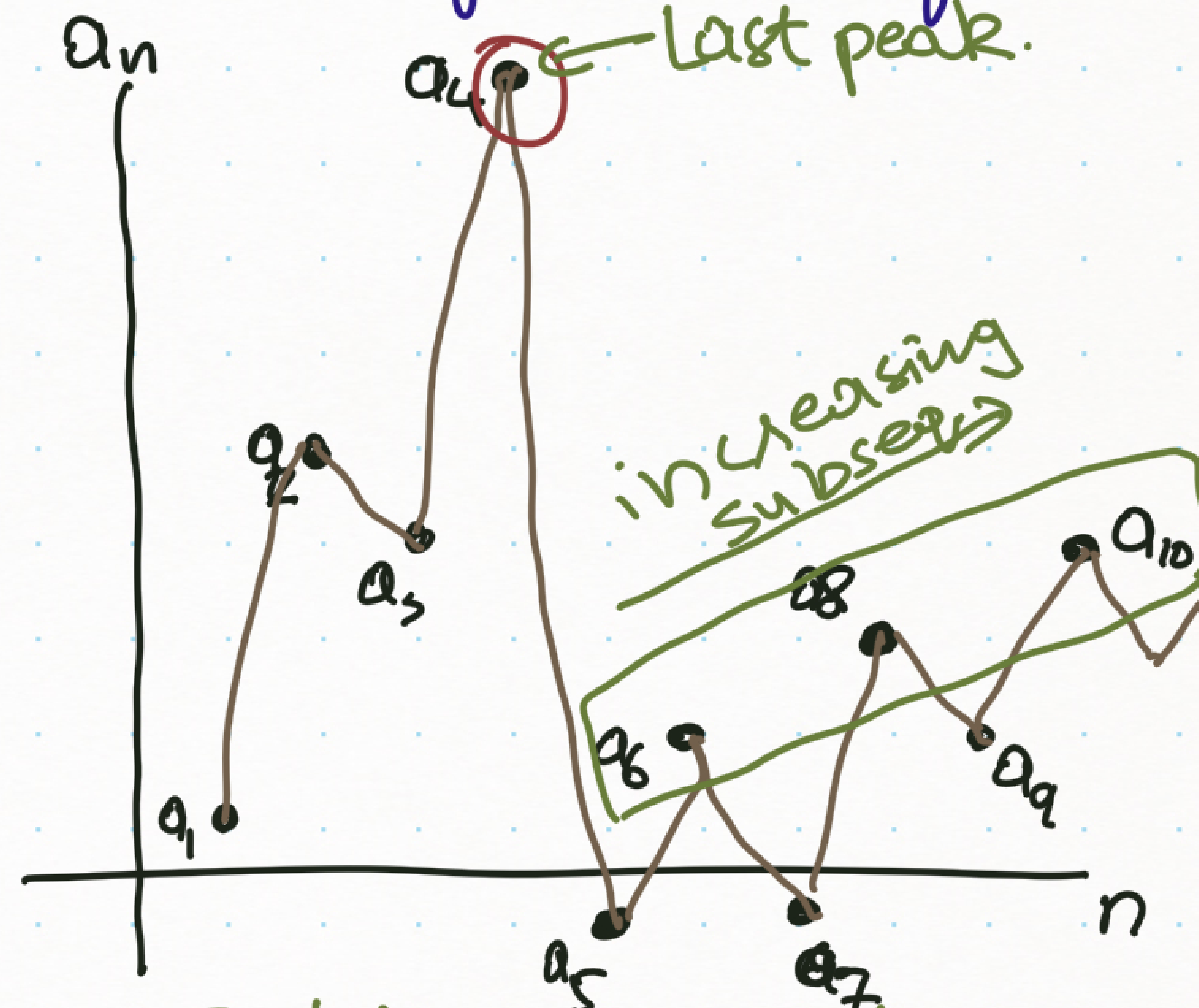

Finitely many peaks.

Lemma Every sequence has a monotone subsequence.

Proof Call a_n a peak if $a_n \geq a_m \ \forall m \geq n$.

a_n is atleast as large as every term after it

Two possibilities →

Case 1 Infinitely many peaks Let a_{n_k} be the k^{th} peak

By defn. of peak, $a_{n_k} \geq a_{n_{k+1}} \ \forall k$ $\therefore$ (a_{n_k}) is a dec. subseq.

Case 2 Finitely many peaks

Lemma Every sequence has a monotone subsequence.

Proof Call a_n a <u>peak</u> if $a_n \geq a_m \ \forall m \geq n$.

Two possibilities ⟶

Case 1 Infinitely many peaks Let a_{n_k} be the k^{th} peak.

By defn. of peak, $a_{n_k} \geq a_{n_{k+1}} \ \forall k$ $\therefore (a_{n_k})$ is a dec. subseq.

Case 2 Finitely many peaks Let a_N be the last peak.

Let $a_{n_1} = a_{N+1}$. (after the last peak) Since a_N was the last peak, a_{N+1} is not a peak & hence $\exists\, a_{n_2}$ after it that's larger than it. Then $a_{n_2} > a_{n_1}$ for some $n_2 > n_1$.

Now a_{n_2} is also not a peak. Hence, there is $a_{n_3} > a_{n_2}$ with $n_3 > n_2$.

This process gives us $a_{n_1} < a_{n_2} < a_{n_3} < \dots$, where $n_1 < n_2 < n_3 < \dots$

Inc. subseq.

Proof of B-W Theorem [(a_n) bdd. $\Rightarrow \exists$ subseq. convergent]

Suppose (a_n) is a bdd. seq.

By Lemma, (a_n) has a montone subseq. (a_{n_k}).

And (a_{n_k}) is bounded (since (a_n) is bounded)

By MCT, (a_{n_k}) is convergent.

$\therefore$ we found a convergent subseq. of (a_n).

$\blacksquare$