Math 400

Real Analysis

Part #15

- Definition of convergence requires knowledge of the limit of the sequence. $|a_n - \underline{a}| < \epsilon$

- MCT is useful because it allows us to show convergence of a sequence without knowing its limit.

  But MCT is only a sufficient condition for convergence.

→ Can we characterize convergence without knowing the limit?

- Definition of convergence requires knowledge of the limit of the sequence.
- MCT is useful because it allows us to show convergence of a sequence without knowing its limit.

  But MCT is only a sufficient condition for convergence.

→ Can we characterize convergence without knowing the limit?

Maybe it means "each term is getting closer to the previous term"

i.e. $\boxed{|a_{n+1} - a_n| < \epsilon \quad \forall n > N.}$

example $(a_n) = (\sqrt{n})$

$$|a_{n+1} - a_n| = |\sqrt{n+1} - \sqrt{n}| = \sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} \Leftrightarrow \frac{1}{\sqrt{n}} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon^2}$$

Given $\epsilon > 0$, let $N = 1/\epsilon^2$. Then $|a_{n+1} - a_n| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{1/\epsilon^2}} = \epsilon$ for all $n > N$.

$$|a_{n+1} - a_n| < \epsilon \quad \forall n > N$$

example $(a_n) = (\sqrt{n})$

$$|a_{n+1} - a_n| = |\sqrt{n+1} - \sqrt{n}| = \sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}}$$

Given $\epsilon > 0$, let $N = 1/\epsilon^2$ Then $|a_{n+1} - a_n| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{1/\epsilon^2}} = \epsilon$ for all $n > N$.

(a_n) diverges even though each $|a_{n+1} - a_n|$ is small.

How to fix this?

Defn A sequence (a_n) is called Cauchy sequence if $\forall \epsilon > 0 \ \exists N$ s.t. $|a_m - a_n| < \epsilon \ \forall m, n \geq N$.

We want all terms to be close to each other not just consecutive terms.

Defn A sequence (a_n) is called <u>Cauchy sequence</u> if $\forall \epsilon > 0 \ \exists N$ s.t. $|a_m - a_n| < \epsilon \ \forall m,n \geq N$.

e.g. $(7 + \frac{1}{n})$ is Cauchy.

$$|a_m - a_n| = \left|\left(7+\frac{1}{m}\right) - \left(7 + \frac{1}{n}\right)\right| = \left|\frac{1}{m} + \frac{-1}{n}\right| \leq \left|\frac{1}{m}\right| + \left|\frac{-1}{n}\right| \quad (\text{by } \Delta \text{ ineq.})$$

$$= \left|\frac{1}{m}\right| + \left|\frac{1}{n}\right| = \frac{1}{m} + \frac{1}{n}$$

Given $\epsilon > 0$, since $\epsilon/2 > 0 \ \exists N > 0$ s.t. $\frac{1}{N} < \frac{\epsilon}{2}$ (by Archimedean Principle)

For any $n, m > N$, $|a_m - a_n| \leq \left|\frac{1}{m}\right| + \left|\frac{1}{n}\right| < \frac{1}{N} + \frac{1}{N}$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$= \epsilon.$$

Lemma If (a_n) is Cauchy then (a_n) is bounded

Proof The Proof is essentially a repetition of the proof we did earlier for "If (a_n) is convergent then (a_n) is bounded".

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Proof The Proof is essentially a repetition of the proof we did earlier for "If (a_n) is convergent then (a_n) is bounded".

Given $\epsilon = 1$, $\exists N$ s.t. $|a_m - a_n| < 1 \;\; \forall m,n \geq N$. Take $m=N$

So in particular, we have $|a_n - a_N| < 1 \;\; \forall n \geq N$

i.e., $a_N - 1 < a_n < a_N + 1 \;\; \forall n \geq N$

i.e., $|a_n| < |a_N| + 1 \;\; \forall n \geq N$

Set $M = \max\{|a_1|, |a_2|, |a_3|, \dots, |a_{N-1}|, |a_N|+1\}$

Then $|a_n| \leq M \;\; \forall n$.

Theorem [Cauchy Criterion]

A sequence converges if and only if it is Cauchy.

Proof $\boxed{\Rightarrow}$ Standard argument left as exercise.

Hint: To show $|a_m - a_n| < \epsilon$

write $|a_m - a_n| = |a_m - a + a - a_n|$ (where $\lim a_n = a$)

$$\le \; ?$$

$$< \; ?$$

$$= \; \epsilon$$

Theorem [Cauchy Criterion]

A sequence converges if and only if it is Cauchy.

Proof $\boxed{\Leftarrow}$ Idea (a_n) Cauchy & hence bounded

Apply Bolzano-Weierstrass to get

$(a_{n_k}) \to a$ cgt. subsequence

Then show $(a_n) \to a$ also.

using defn. of Cauchy.

Theorem [Cauchy Criterion]

A sequence converges if and only if it is Cauchy.

Proof $\boxed{\Leftarrow}$ Let (a_n) be Cauchy. By Lemma, (a_n) is bounded.

B-W $\Rightarrow$ $\exists$ a convergent subsequence (a_{n_k}).

Let $a = \lim a_{n_k}$

Given $\epsilon > 0$, since (a_n) is Cauchy, $\exists N_1$ s.t. $|a_n - a_m| < \frac{\epsilon}{2}$ $\forall n, m \geq N_1$ — (1)

Since $(a_{n_k}) \to a$, $\exists N_2$ s.t. $|a_{n_k} - a| < \frac{\epsilon}{2}$ $\forall n_k > N_2$. — (2)

Set $N = \max\{N_1, N_2\}$. Then for any $m \geq N$,

$|a_m - a| = \ldots\ldots < \epsilon$

Theorem [Cauchy Criterion]

A sequence converges if and only if it is Cauchy.

Proof $\boxed{\Leftarrow}$ Let (a_n) be Cauchy. By Lemma, (a_n) is bounded.

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Let $a = \lim a_{n_k}$

Given $\epsilon > 0$, since (a_n) is Cauchy, $\exists N_1$ s.t. $|a_n - a_m| < \frac{\epsilon}{2}$ $\forall n, m \geq N_1$ — ①

Since $(a_{n_k}) \to a$, $\exists N_2$ s.t. $|a_{n_k} - a| < \frac{\epsilon}{2}$ $\forall n_k > N_2$. — ②

Set $N = \max\{N_1, N_2\}$. Then for any $m \geq N$,

$$|a_m - a| = |a_m - a_{n_N} + a_{n_N} - a| \leq |a_m - a_{n_N}| + |a_{n_N} - a|$$

$$< \epsilon/2 + \epsilon/2$$

by ① since $n_N \geq N \geq N_1$

defn. of subseq..

By ② since $n_N \geq N \geq N_2$

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Part #16

Recall For an infinite series $\sum_{k=1}^{\infty} a_k$

- the sequence of terms is $(a_1, a_2, a_3, \ldots)$
- the sequence of partial sums is $(s_1, s_2, s_3, \ldots)$
  where $s_m = a_1 + a_2 + a_3 + \ldots + a_m$.
- $\sum_{k=1}^{\infty} a_k$ is convergent $\Longleftrightarrow$ (s_m) is convergent
- If convergent, $\sum_{k=1}^{\infty} a_k = A \Leftrightarrow \lim_{m\to\infty} s_m = A$

Theorem [Algebra of Series Limits]

If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

(i) $\sum_{k=1}^{\infty} (ca_k) = cA$ for all $c \in \mathbb{R}$,

(ii) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$

Proof Simple application of Algebra of Sequence limits.

$$\sum_{k=1}^{\infty}(a_k+b_k) \overset{①}{=} \lim_{m\to\infty}\left(\sum_{k=1}^{m}(a_k+b_k)\right) \overset{②}{=} \lim_{m\to\infty}\left(\left(\sum_{k=1}^{m}a_k\right)+\left(\sum_{k=1}^{m}b_k\right)\right)$$

$$\overset{③}{=} \lim_{m\to\infty}\left(\sum_{k=1}^{m}a_k\right) + \lim_{m\to\infty}\left(\sum_{k=1}^{\infty}b_k\right) \overset{④}{=} A + B$$

What are the reasons for each step ①, ②, ③, ④ ?

Theorem [Cauchy Criterion for Series]

$\sum_{k=1}^{\infty} a_k$ converges $\Longleftrightarrow$ $\forall \epsilon > 0, \exists N$ s.t.

$$\boxed{|a_{m+1} + a_{m+2} + \ldots\ldots + a_n| < \epsilon} \quad \forall n > m \geq N$$

Proof $|S_n - S_m| = |a_{m+1} + a_{m+2} + \ldots + a_n|$ $\longleftarrow$ Apply Cauchy criterion for seq.

$\sum a_k$ convergent $\Longleftrightarrow$ (S_m) is convergent

$\Longleftrightarrow$ (S_m) is a Cauchy seq.

$\Longleftrightarrow$ $\forall \epsilon > 0\ \exists N$ s.t. $|S_m - S_n| < \epsilon \quad \forall n, m \geq N$

Theorem [Cauchy Criterion for Series]

$\sum_{k=1}^{\infty} a_k$ converges $\iff$ $\forall \epsilon > 0, \exists N$ s.t.

$|a_{m+1} + a_{m+2} + \ldots\ldots + a_n| < \epsilon \quad \forall n > m \geq N$

$|a_n| < \epsilon$ i.e, $|a_n - 0| < \epsilon$

Cor If $\sum_{k=1}^{\infty} a_k$ converges then $(a_k) \to 0$

Proof Apply Cauchy criterion for series with $n = m+1$.

Theorem [Comparison Test]

Suppose the sequences (a_n) & (b_n) satisfy $0 \leq a_k \leq b_k \ \forall k$

(i) If $\sum_{k=1}^{\infty} b_k$ converges then

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges then

$$\sum \frac{1}{\log n + 1} \geq \sum \frac{1}{n} \geq \sum \frac{1}{n^2 - n} \geq \sum \frac{1}{n^2} \geq \sum \frac{1}{n^2 + n}$$

? Dgt. ? Cgt. ?

Theorem [Comparison Test]

Suppose the sequences (a_n) & (b_n) satisfy $0 \le a_k \le b_k \ \forall k$

(i) If $\sum_{k=1}^{\infty} b_k$ converges then $\sum_{k=1}^{\infty} a_k$ converges $\quad \sum 0 \le \sum a_k \le \sum b_k$

cgt. cgt

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges then $\sum_{k=1}^{\infty} b_k$ diverges

Proof Apply Cauchy Criterion for series using

$$|a_{m+1} + a_{m+2} + \dots + a_n| \le |b_{m+1} + b_{m+2} + \dots + b_n| < \epsilon$$

Recall $\rightarrow$

[Series p-test] $\sum_{n=1}^{\infty} 1/n^p$ converges $\Longleftrightarrow$ $p > 1$.

Proof ① If $p \leq 1$ then $\frac{1}{n} \leq \frac{1}{n^p}$ $\forall n$

Use this to show $\sum_{n=1}^{\infty} 1/n^p$ is divergent (Using Comp. T. with $\sum \frac{1}{n}$ dgt.)

② If $p > 1$ then use Cauchy Condensation Test.

[Geometric Series] A Geometric Series is of the form:

$$\sum_{k=0}^{\infty} ar^k = \underbrace{a + ar + ar^2 + \cdots}_{\text{is convergent} \iff |r|<1}, \quad \text{for some fixed } a \,\&\, r.$$

& then the sum $= \dfrac{a}{1-r}$

If $a=0$ then sum $=0$

If $a \neq 0$ and $r=1$ then divergent

If $\underline{a \neq 0 \text{ and } r \neq 1}$ then $\underline{S_m = a + ar + \cdots + ar^{m-1}}$ seq. of partial sums

$$= \frac{a(1-r^m)}{1-r}$$

using the identity

$$(1-r)(1+r+r^2+\cdots+r^{m-1}) = 1-r^m$$

$$\underline{\lim_{m\to\infty} S_m} = \underline{\lim_{m\to\infty} \frac{a(1-r^m)}{1-r}} = \left[\frac{a}{1-r}\right]\left(\lim_{m\to\infty}(1-r^m)\right)$$

exists $\iff -1<r<1$

$$= \left[\frac{a}{1-r}\right)\left(1 - \lim_{m\to\infty} r^m\right) = \frac{a}{1-r} \iff |r|<1$$

limit exists $\iff |r|<1$

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Part #17

What if our series contains negative terms also?

Theorem [Absolute Convergence Test]

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges as well.

What if our series contains negative terms also?

Theorem [Absolute Convergence Test]

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges as well.

Proof (Apply Cauchy Criterion).

Given $\epsilon > 0$, since $\sum |a_n|$ converges, by Cauchy Criterion

$\exists N$ s.t. $|a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \epsilon \quad \forall n > m \geq N$.

What if our series contains negative terms also?

Theorem [Absolute Convergence Test]

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges as well.

Proof (Apply Cauchy Criterion).

Given $\epsilon > 0$, since $\sum |a_n|$ converges, by Cauchy Criterion $\exists N$ s.t. $|a_{m+1}| + |a_{m+2}| + \cdots + |a_n| < \epsilon \quad \forall n > m \geq N$.

By Δ-inequality,

$|a_{m+1} + \cdots + a_n| < |a_{m+1}| + \cdots + |a_n| < \epsilon \quad \forall n > m \geq N$

Hence by Cauchy Criterion, $\sum a_n$ converges.

Theorem [Alternating Series Test]

Let (a_n) satisfy (i) $a_1 \geq a_2 \geq \cdots\cdots \geq a_n \geq a_{n+1} \geq \cdots$, and

(ii) $(a_n) \to 0$

$(a_n) \downarrow 0$

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Theorem [Alternating Series Test]

Let (a_n) satisfy (i) $a_1 \geq a_2 \geq \cdots\cdots \geq a_n \geq a_{n+1} \geq \cdots$, and

(ii) $(a_n) \to 0$

$(a_n) \downarrow 0$

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof [HW Exercise]

seq. of partial sums

Proof #1 $\to$ Show (S_m) is a Cauchy seq.

subseq. of (S_m)

Proof #2 $\to$ Show subsequences (S_{2m}) and (S_{2m+1}) are both convergent by MCT & that implies (S_m) is also convergent.

example $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ (alternating Harmonic series)

is convergent by Alternating Series Test.

Do you know the value of the limit?

But $\sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{1}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ (Harmonic series) is divergent.

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Do you know the value of the limit?

But $\sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{1}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ (Harmonic series) is divergent.

Definition If $\sum_{k=1}^{\infty} |a_k|$ converges then we say $\sum_{k=1}^{\infty} a_k$ converges absolutely.

If $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges then we say $\sum_{k=1}^{\infty} a_k$ converges conditionally.

Examples for each type?

A rearrangement of a series is obtained by permuting the terms in the sum into some other order

- all the original terms eventually appear
- no original term is repeated
- no new terms are introduced.

Defn A rearrangement of $\sum_{k=1}^{\infty} a_k$ is a series $\sum_{k=1}^{\infty} b_k$ for which there is a bijection $f: \mathbb{N} \to \mathbb{N}$ s.t. $b_{f(k)} = a_k \ \forall k$.

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots\cdots$

$\frac{1}{5} + \frac{1}{3} - \frac{1}{2} - \frac{1}{6} + 1 + \frac{1}{10871} + \frac{1}{7} - \frac{1}{4} + \cdots\cdots$ rearrangement

In the very first video we used rearrangement of series to show how that can lead to contradictions.

Look at $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$

we know its convergent & hence equals some number S.

Then, $\frac{1}{2}S = 0 \quad \frac{1}{2} \quad 0 \quad -\frac{1}{4} \quad 0 \quad +\frac{1}{6} \quad 0 \quad -\frac{1}{8} \quad +\frac{1}{10} \cdots\cdots$

add to

$$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \cdots\cdots$$

to get

$$\frac{3}{2}S = 1 \quad + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \quad + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} \quad + \frac{1}{11} \cdots\cdots$$

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Look at $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\cdots$

we know its convergent & hence equals some number S.

Then, $\frac{1}{2} S = 0 \quad \frac{1}{2} \quad 0 \quad -\frac{1}{4} \quad 0 \quad +\frac{1}{6} \quad 0 \quad -\frac{1}{8} \quad +\frac{1}{10} \cdots\cdots$

add to

$S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \cdots\cdots$

to get

$\frac{3}{2} S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} \cdots\cdots$

$\frac{3}{2} S = S$ ✗

Rearrangement of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$ Two positive terms followed by negative...

[Rearrangement theorem I]

If $\sum_{k=1}^{\infty} a_k$ converges conditionally then

for any L ($L \in \mathbb{R}$ or $L = \pm\infty$)

$\exists$ rearrangement of $\sum_{k=1}^{\infty} a_k$ that converges to L

[Rearrangement Theorem I]

If $\sum_{k=1}^{\infty} a_k$ converges conditionally, then

for any L ($L \in \mathbb{R}$ or $L = \pm\infty$)

$\exists$ rearrangement of $\sum_{k=1}^{\infty} a_k$ that converges to L

[Rearrangement Theorem II]

If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then

every rearrangement of this series converges to the same limit.

Idea for RT-I

Suppose $L \in \mathbb{R}^+$

Since $\sum a_n$ converges conditionally it must have both positive and negative terms.

Let p_k be the k^{th} positive term of (a_n)

n_k be the k^{th} negative term of (a_n)

- $\sum a_n$ converges $\Rightarrow (a_n) \to 0$
- $\sum a_n$ converges conditionally then $\sum_{k=1}^{\infty} p_k = \infty$ (while $(p_k) \to 0$)

  & $\sum_{k=1}^{\infty} n_k = -\infty$ (while $(n_k) \to 0$)

Idea for RT-I

Suppose $L \in \mathbb{R}^+$

Since $\sum a_n$ converges conditionally it must have both positive and negative terms.

Let p_k be the k^{th} positive term of (a_n)

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- $\sum a_n$ converges $\Rightarrow (a_n) \to 0$
- $\sum a_n$ converges conditionally then $\sum_{k=1}^{\infty} p_k = \infty$ (while $(p_k) \to 0$) & $\sum_{k=1}^{\infty} n_k = -\infty$ (while $(n_k) \to 0$)

- $\sum p_k = \infty \Rightarrow \exists P_1$ s.t. $\sum_{k=1}^{P_1} p_k > L$
- $\sum n_k = -\infty \Rightarrow \exists N_1$ s.t. $\sum_{k=1}^{P_1} p_k + \sum_{k=1}^{N_1} n_k < L$

Repeat

Idea for RT-I

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Since $\sum a_n$ converges conditionally it must have both positive and negative terms.

Let p_k be the k^{th} positive term of (a_n)

n_k be the k^{th} negative term of (a_n)

- $\sum a_n$ converges $\Rightarrow (a_n) \to 0$
- $\sum a_n$ converges conditionally then $\sum_{k=1}^{\infty} p_k = \infty$ (while $(p_k) \to 0$)

  & $\sum_{k=1}^{\infty} n_k = -\infty$ (while $(n_k) \to 0$)

- $\sum p_k = \infty \Rightarrow \exists P_1$ s.t. $\sum_{k=1}^{P_1} p_k > L$
- $\sum n_k = -\infty \Rightarrow \exists N_1$ s.t. $\sum_{k=1}^{P_1} p_k + \sum_{k=1}^{N_1} n_k < L$

Repeat $\sum_{k=1}^{P_1} p_k + \sum^{N_1} n_k + \sum^{P_2} p_k > L \longrightarrow \sum^{P_1} p_k + \sum^{N_1} n_k + \sum^{P_2} p_k + \sum^{N_2} n_k < L$

$\longrightarrow \cdots\cdots$