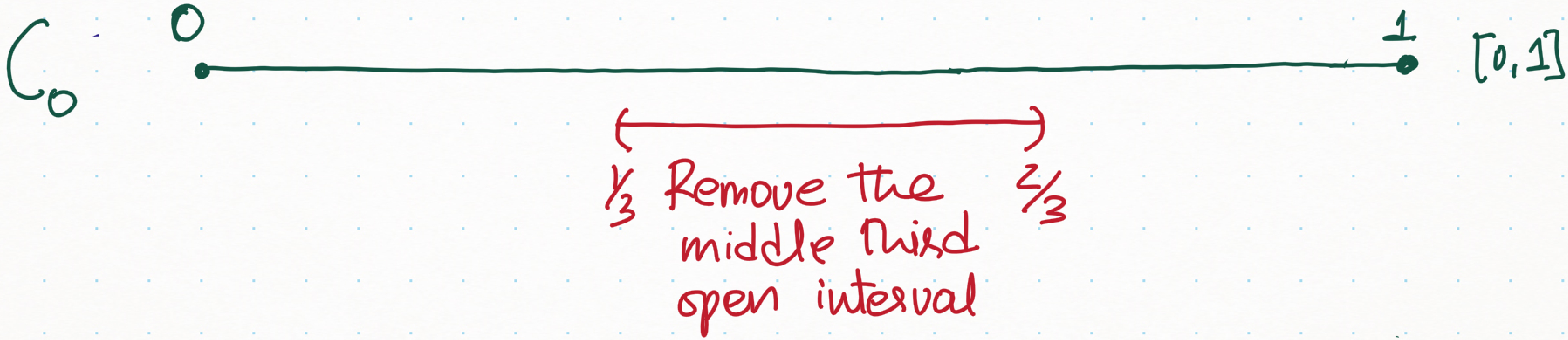
Math 400

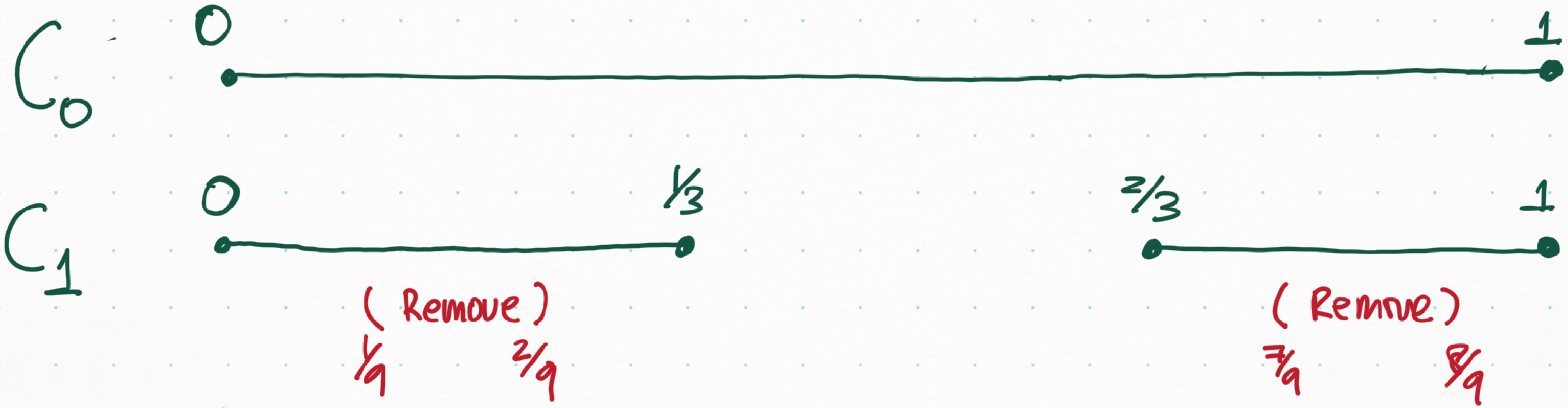
Real Analysis

Part #18

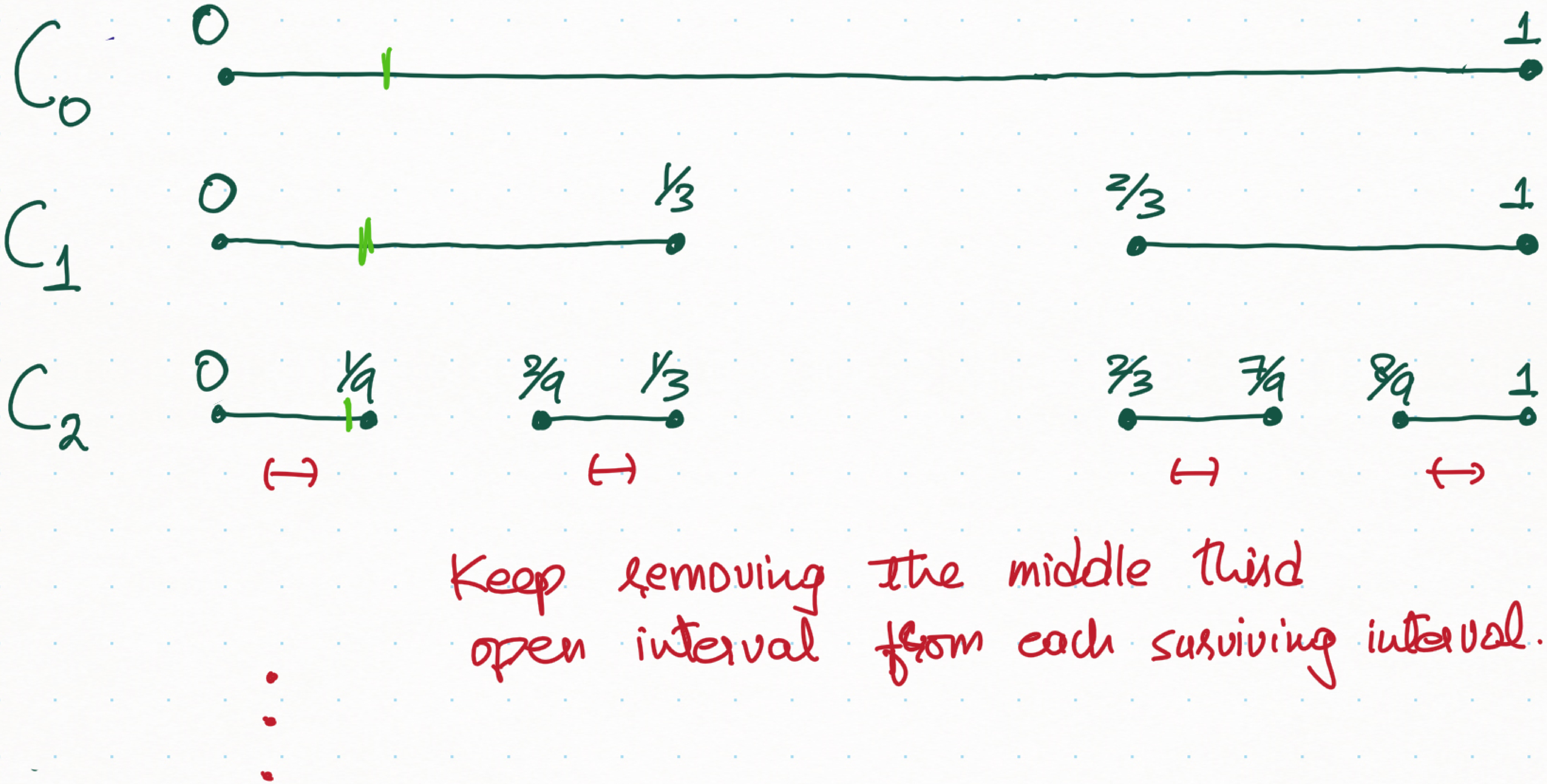
The Cantor Set



The Cantor Set



The Cantor Set



Keep removing the middle third open interval from each surviving interval.

C_n will consist of 2^n closed intervals each of length $1/3^n$ for $n = 0, 1, 2, \ldots\ldots$

Defn Cantor Set $C = \bigcap_{n=0}^{\infty} C_n$

In other words, $C = [0,1] \setminus \underbrace{\left[\left(\frac{1}{3}, \frac{2}{3}\right) \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left(\frac{7}{9}, \frac{8}{9}\right) \cup \ldots\right]}_{\text{intervals that were removed}}$

Ques Is $C \neq \emptyset$?

- $C \neq \phi$ $\quad 0, 1 \in C_n \ \forall n$

In fact, endpoints of the closed intervals are never removed.

Ques How "big" is C?

• $C \neq \phi$ $\quad$ $0, 1 \in C_n \ \forall n$

In fact, endpoints of the closed intervals are never removed.

Ques How "big" is C?

→ Cardinality

→ Length

→ Dimension

Cardinality C is uncountable!

We can find a bijection between C and all binary sequences $\{(a_n)_{n=1}^{\infty} : a_n = 0 \text{ or } 1\}$

For $c \in C$, $a_1 = 0$ if c belongs to the left-hand interval of C_1
$a_1 = 1$ if ——— " ——— right-hand ——— " ———

Based on $a_1 \to$ $a_2 = 0$ if c belongs to the left-hand interval of the component of C_2 (as indicated by a_1)
$a_2 = 1$ if ——— " ——— right-hand ——— " ———
——— " ———

⋮

Every point yields a sequence and every sequence defines a point.

Length $\quad C$ has length zero !

$$\text{length of } C = \text{length of } [0,1]$$
$$\qquad - \text{length of all the removed intervals}$$
$$= 1 - \left(\frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots + 2^{n-1}\left(\frac{1}{3^n}\right) + \cdots\right)$$

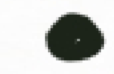
$$= 1 - \left(\frac{1/3}{1-2/3}\right)$$

$$= 1 - 1$$

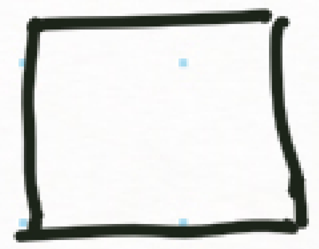
$$= 0$$

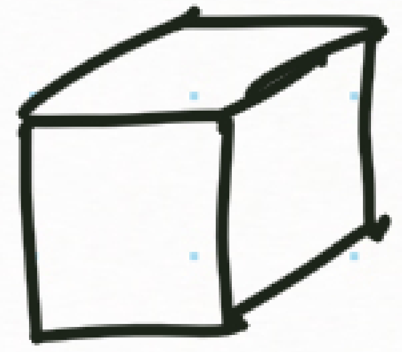
Dimension $\quad \dim(C) = 0.631\ldots$ (!!)

we all agree dimension of

 is zero

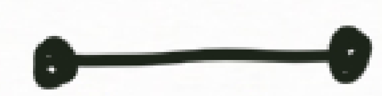
 is one

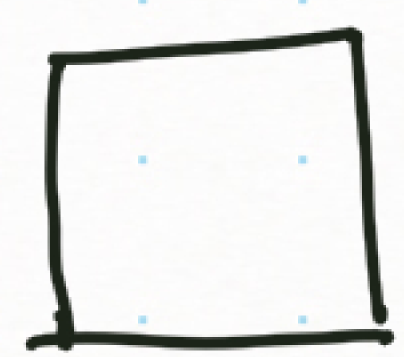
 is two

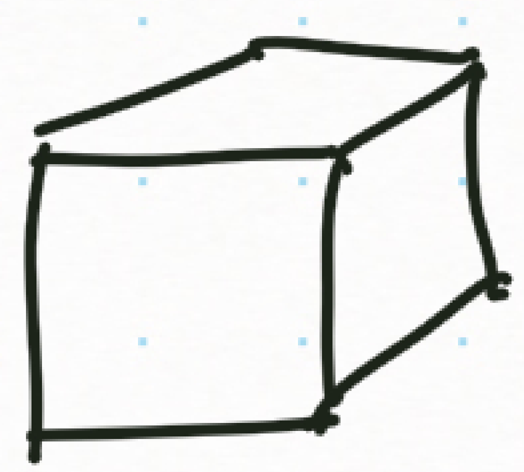
 is three

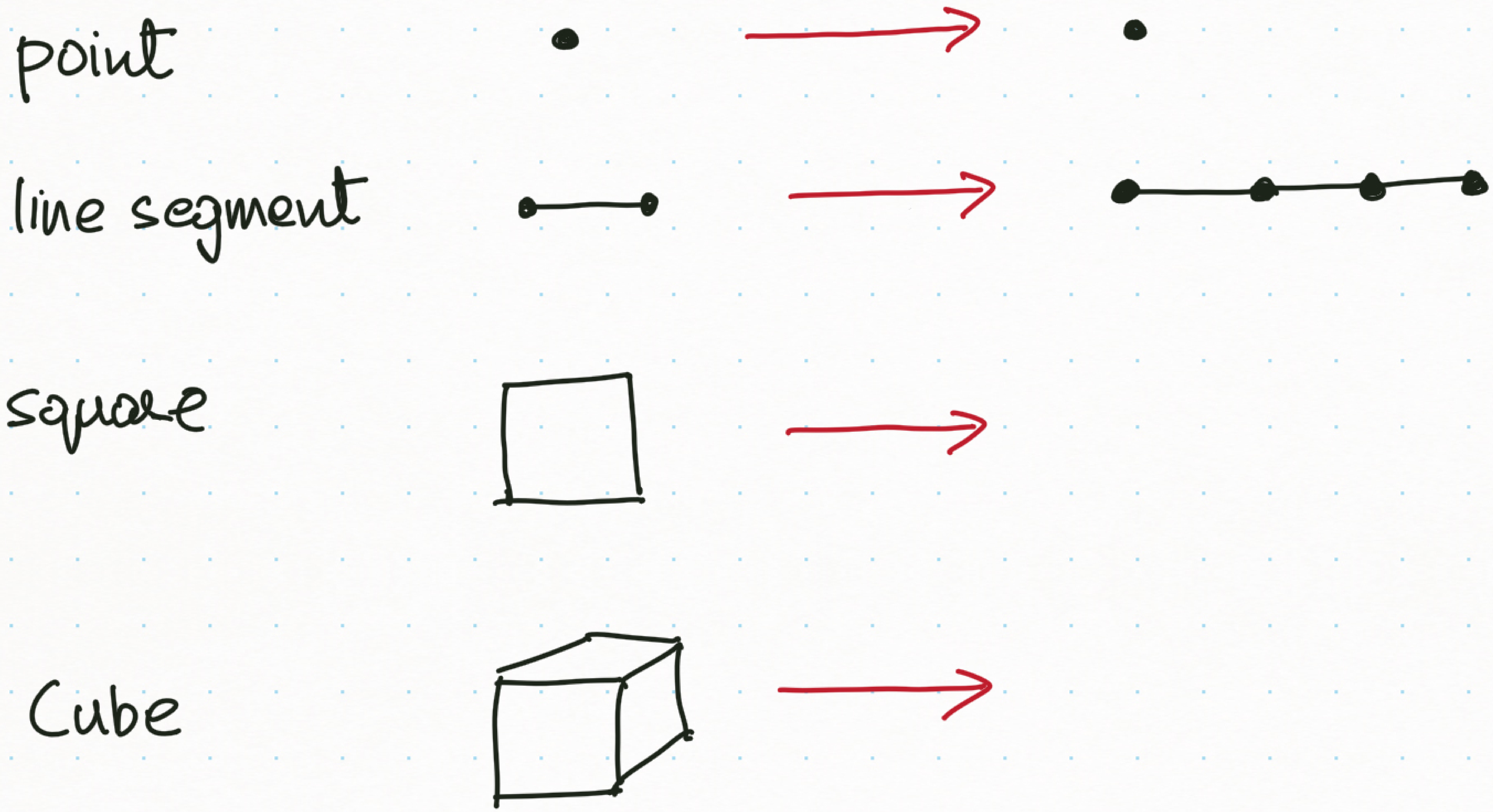
What happens when we magnify each such set by a factor of 3?

point 

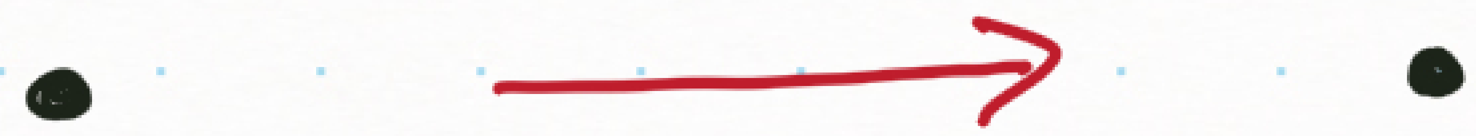
line segment 

square 

Cube 

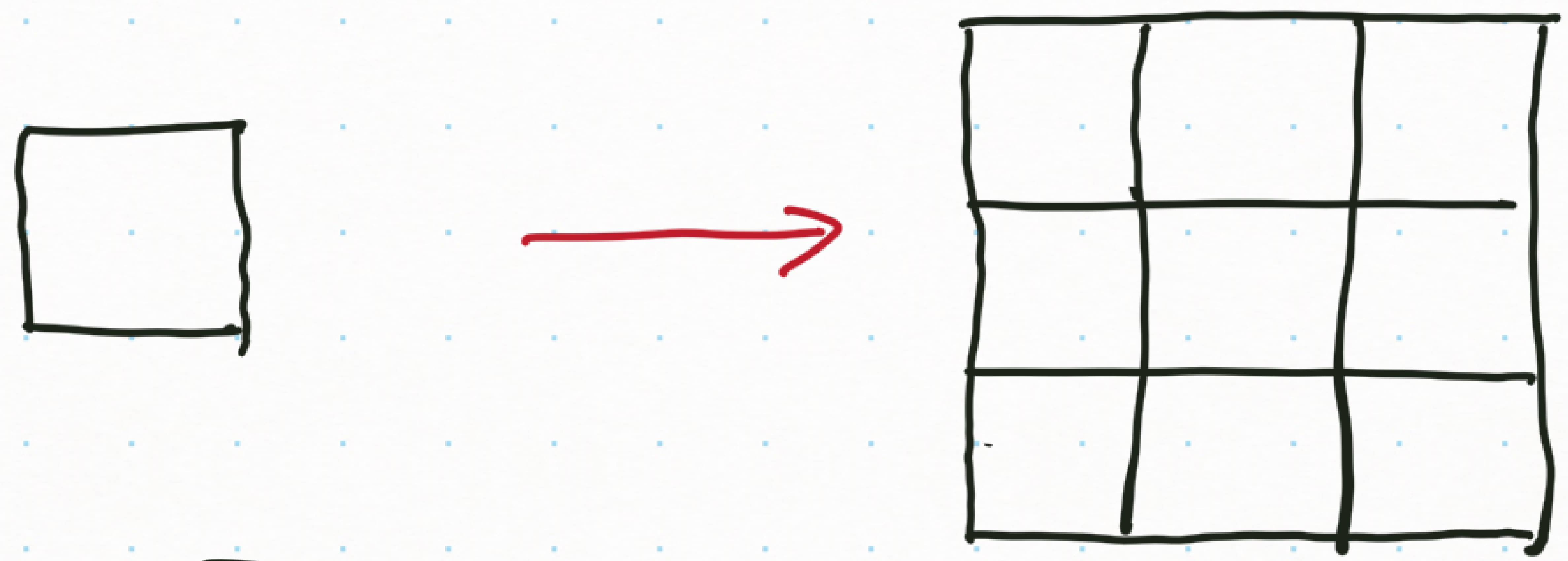

point
line segment
square
Cube

point

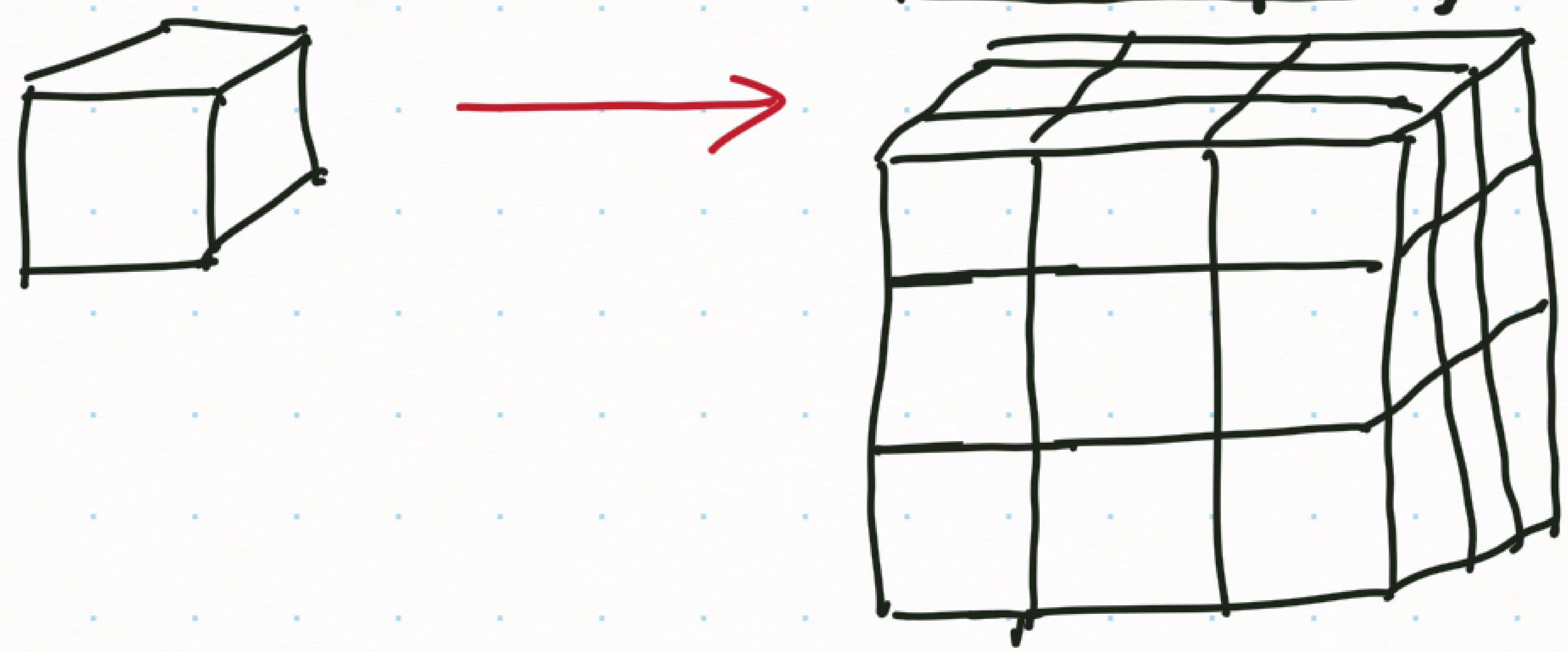


line segment

square



Cube

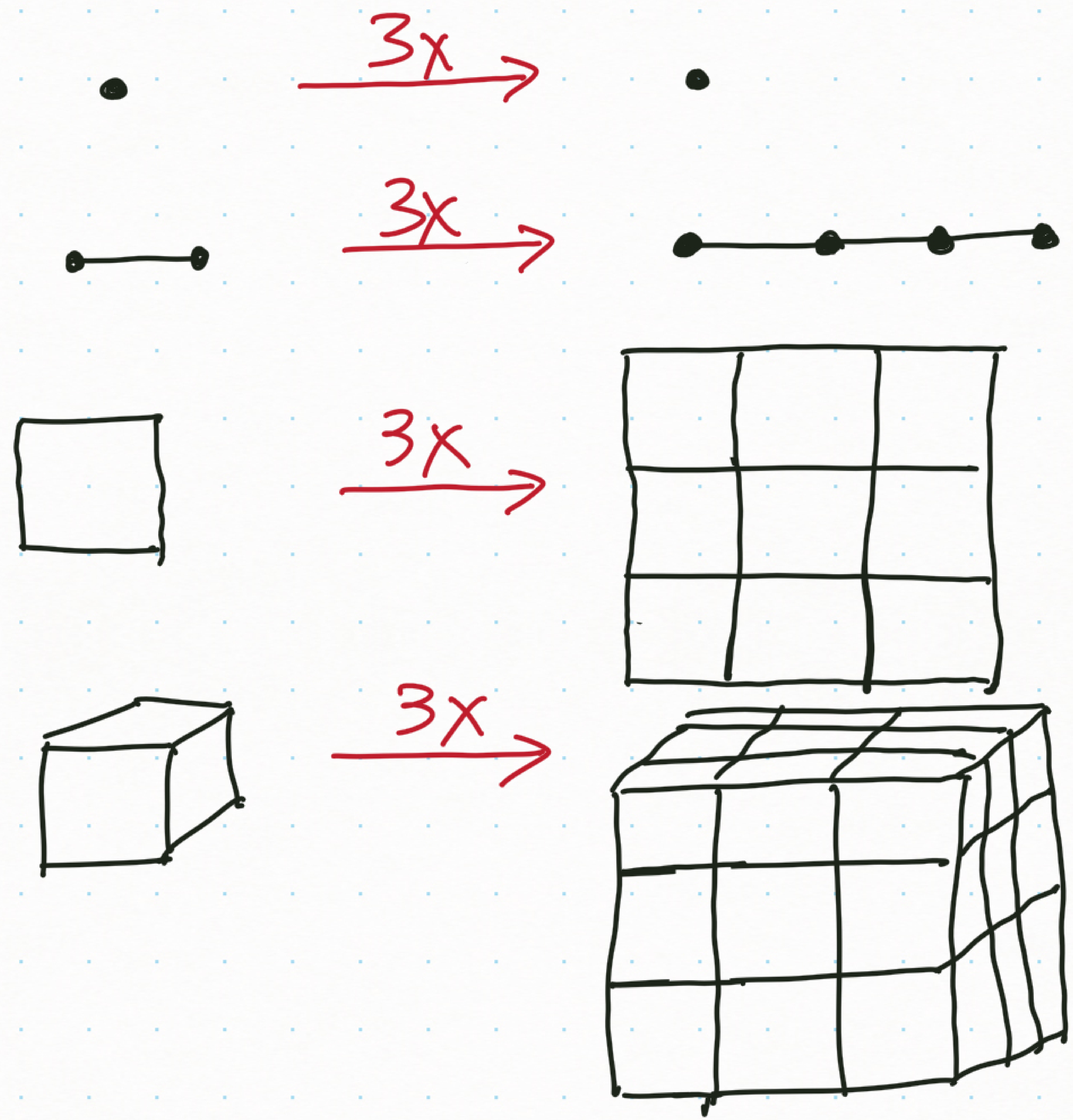


point

line segment

square

Cube



$1 = 3^0$ copies

$3 = 3^1$ copies

$9 = 3^2$ copies

$27 = 3^3$ copies

$3^{\text{dimension}}$

How many 3^d copies of C are created?

$C_0 = [0, 1]$ becomes $[0, 3]$

and C_1 becomes $[0, 1] \cup [2, 3]$

$$\underset{\text{copy of } C}{\underbrace{[0,1]}} \cup \underset{\text{copy of } C}{\underbrace{[2,3]}}$$

$\therefore C$ magnifies to $\underline{2}$ copies of C.

$$\underline{2 = 3^d} \iff d = \frac{\log 2}{\log 3} \simeq 0.631\ldots$$

Math 400

Real Analysis

Part #19

Recall For $a \in \mathbb{R}$ and $\epsilon > 0$
ϵ-neighborhood of a is $V_\epsilon(a) = \{x \in \mathbb{R} : |x-a| < \epsilon\}$
$= (a-\epsilon, a+\epsilon)$

Defn Set $\mathcal{O} \subseteq \mathbb{R}$ is open if $\forall a \in \mathcal{O}\ \exists \epsilon > 0$ s.t.
$V_\epsilon(a) \subseteq \mathcal{O}$

Examples

Recall For $a \in \mathbb{R}$ and $\epsilon > 0$
ϵ-neighborhood of a is $V_\epsilon(a) = \{x \in \mathbb{R} : |x-a| < \epsilon\}$
$= (a-\epsilon, a+\epsilon)$

Defn Set $O \subseteq \mathbb{R}$ is open if $\forall a \in O \ \exists \epsilon > 0$ s.t.
$V_\epsilon(a) \subseteq O$

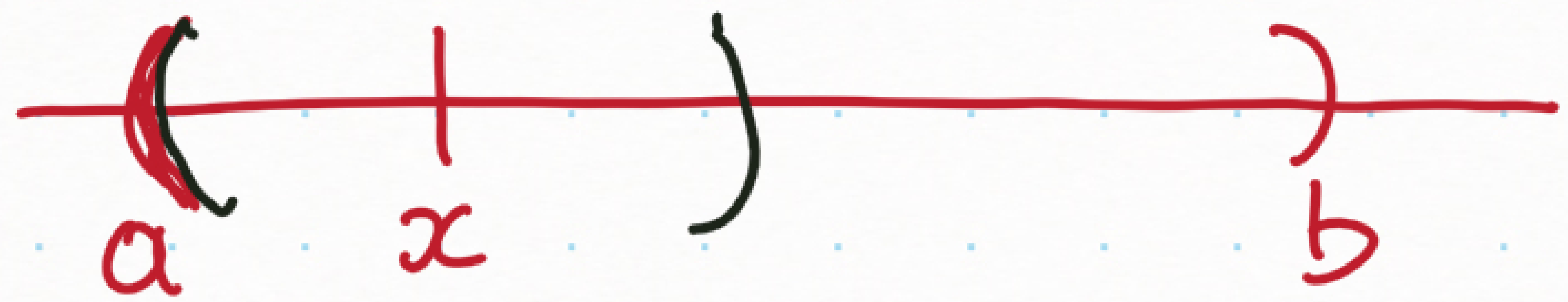
Examples

- $\mathbb{R}$ is open Let $a \in \mathbb{R}$ Then pick $\epsilon = 1$
  & $V_1(a) = (a-1, a+1) \subseteq \mathbb{R}$
- $\emptyset$ is open

- Open interval (a,b) is open

Let $x \in (a,b)$ Pick $\epsilon = ?$ so that $V_\epsilon(x) = (x-\epsilon, x+\epsilon)$

$\subseteq (a,b)$

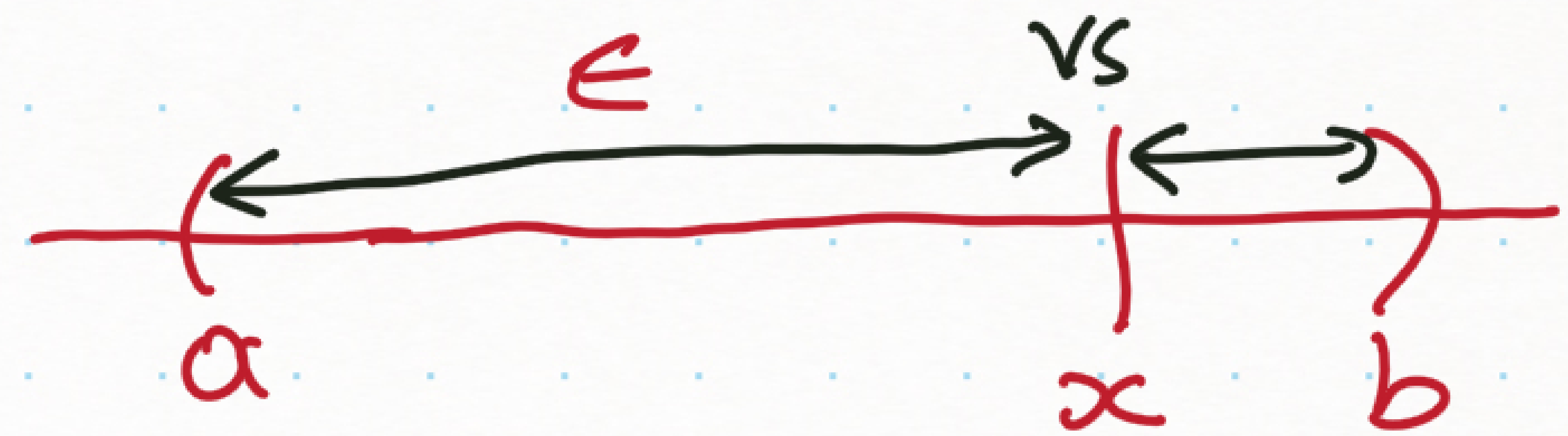


- Open interval (a,b) is open

Let $x \in (a,b)$ Pick $\epsilon = ?$ so that $V_\epsilon(x) = (x-\epsilon, x+\epsilon)$
$\subseteq (a,b)$

Pick $\epsilon = x - a$?

But what if



?

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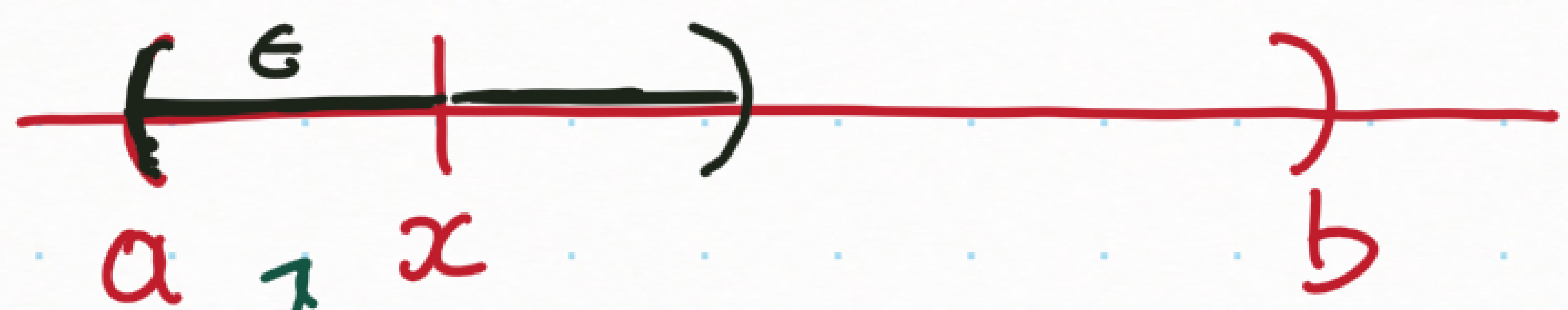

Pick $\epsilon = x - a$?

But what if

?

Check!

$$\boxed{\begin{array}{l}\text{Pick } \underline{\epsilon = \min\{x-a,\, b-x\}} \\ \text{then } V_\epsilon(x) = (x-\epsilon, x+\epsilon) \subseteq (a,b)\end{array}}$$

- Intervals (a,∞) and $(-\infty,b)$ are open

What ϵ should we use in each case?

- Intervals (a,∞) and $(-\infty,b)$ are open

What ϵ should we use in each case?

- $[2,5]$ is not an open set

We have to argue: $\underbrace{\exists x \in [2,5] \text{ s.t. } V_\epsilon(x) = (x-\epsilon, x+\epsilon) \not\subseteq [2,5] \quad \forall \epsilon > 0}_{\text{negation of defn. of open set}}$

What $x \in [2,5]$ should we use? $x = 4$?

- Intervals (a,∞) and $(-\infty,b)$ are open

What ϵ should we use in each case?

- $[2,5]$ is not an open set

We have to argue: $\exists\, x \in [2,5]$ s.t. $V_\epsilon(x) = (x-\epsilon, x+\epsilon) \not\subseteq [2,5]$ $\forall\, \epsilon > 0$.

negation of defn. of open set

What $x \in [2,5]$ should we use? $x=4$? ~~X~~

Pick $x=2$ Verify $(2-\epsilon, 2+\epsilon) \not\subseteq [2,5]$

Yes, $2-\epsilon < 2$

any index set
e.g. $\{1,2,\ldots,k\}$, $\mathbb{N}$, $\mathbb{R}$, $\mathcal{P}(\mathbb{R})$, ...

Theorem ① If $\{O_\lambda : \lambda \in \Lambda\}$ is any collection of open sets then $\bigcup_{\lambda \in \Lambda} O_\lambda$ is also open.

② If $\{O_1, O_2, \ldots O_k\}$ is a finite collection of open sets then $\bigcap_{i=1}^{k} O_i$ is also open.

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Proof ① Let $a \in \bigcup_{\lambda\in\Lambda} O_\lambda$. We need an $\epsilon>0$ s.t. $V_\epsilon(a) \subseteq \bigcup_{\lambda\in\Lambda} O_\lambda$

Since $a \in \bigcup_{\lambda\in\Lambda} O_\lambda$, $a \in O_{\lambda'}$ for some $\lambda' \in \Lambda$

$O_{\lambda'}$ is open, so $\exists\ \epsilon>0$ s.t. $V_\epsilon(a) \subseteq O_{\lambda'}$ which is $\subseteq \bigcup_{\lambda\in\Lambda} O_\lambda$ as needed.

any index set
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<u>Proof ②</u> Let $a \in \bigcap_{i=1}^{k} O_i$. Since $a \in O_i \ \forall i$ and each O_i is open, we have $\exists\, \epsilon_1 > 0, \epsilon_2 > 0, \ldots, \epsilon_k > 0$ s.t. $V_{\epsilon_i}(a) \subseteq O_i \ \forall i = 1, \ldots, k$.

But we need one $\epsilon > 0$ s.t. $V_\epsilon(a) \subseteq \bigcap_{i=1}^{k} O_i$ i.e., $V_\epsilon(a) \subseteq O_i \ \forall i = 1, \ldots, k$.

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Let $\underline{\epsilon = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_k\}}$ then $\underline{V_\epsilon(a) \subseteq V_{\epsilon_i}(a)}\ \forall i$, so, $\underline{V_\epsilon(a) \subseteq O_i \ \forall i}$.

Apply the previous theorem to our examples of open sets:
open intervals of the form (a,b), $(-\infty,a)$, (b,∞).
What kind of sets do you get when you take unions or
intersections of those open intervals?

Apply the previous theorem to our examples of open sets: open intervals of the form (a,b), $(-\infty,a)$, (b,∞).

What kind of sets do you get when you take unions or intersections of those open intervals?

You will always get a union of open intervals.

e.g. $\left((-\infty,a)\cup(b,\infty)\right)\cap\left((-\infty,c)\cup(d,\infty)\right)$

$= (-\infty, \min\{a,c\}) \cup (\max\{b,d\},\infty)$

Theorem Every open set is a countable union of disjoint open intervals.

Proof (Outline)

Let $\mathcal{O}$ be an open set

Each $x \in \mathcal{O}$ is contained in $(x-\epsilon, x+\epsilon) \subseteq \mathcal{O}$ for some $\epsilon > 0$

Let I_x be the largest open interval in $\mathcal{O}$ that contains x

??

$I_x = (\alpha, \beta)$ What is α? β?

s.t. $x \in I_x$ ✓

and $I_x \subseteq \mathcal{O}$ ✓

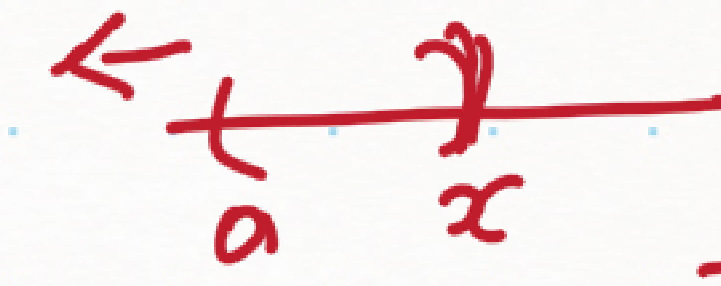
Theorem Every open set is a countable union of disjoint open intervals.

Proof (Outline)

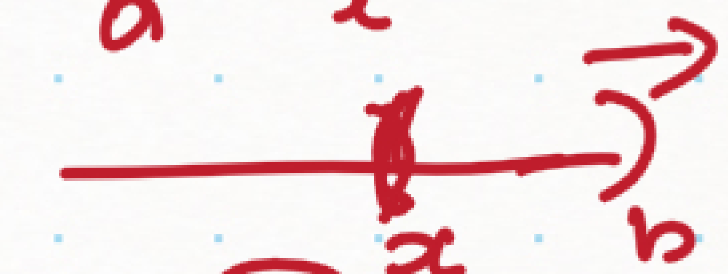
Let $\mathcal{O}$ be an open set

Each $x \in \mathcal{O}$ is contained in $(x-\epsilon, x+\epsilon) \subseteq \mathcal{O}$ for some $\epsilon > 0$

Let $I_x = (\alpha, \beta)$ where $\alpha = \inf\{a : (a, x) \subseteq \mathcal{O}\}$



$\beta = \sup\{b : (x, b) \subseteq \mathcal{O}\}$



So, $\boxed{\mathcal{O} = \bigcup_{x \in \mathcal{O}} I_x}$ But why is this a countable union?

Observation: $\forall x, y \in \mathcal{O}$, $I_x = I_y$ or $I_x \cap I_y = \emptyset$

Since each I_x contains a rational #, Observation tells us there can not be more intervals than $|\mathbb{Q}|$

MATH 400

Real Analysis

Part #20

<u>Defn</u> A point x is a <u>limit point</u> of a set A if $V_\epsilon(x) \cap (A \setminus \{x\}) \neq \emptyset \;\; \forall \epsilon > 0.$

"every ϵ-neighborhood of x intersects A in something other than x."

<u>Another way</u> $\rightarrow$

Defn A point x is a limit point of a set A if $V_\epsilon(x) \cap (A \setminus \{x\}) \neq \phi$ $\forall \epsilon > 0$.

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Another way $\rightarrow$

Theorem x is a limit point of A iff $x = \lim_{n\to\infty} a_n$ for some sequence $(a_n) \subseteq A - \{x\}$

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Theorem: x is a limit point of A iff $x = \lim\limits_{n\to\infty} a_n$ for some sequence $(a_n) \subseteq A - \{x\}$

Ques What are all the limit points of $(-1, 1)$?

Defn A point x is a <u>limit point</u> of a set A if $V_\epsilon(x) \cap (A \setminus \{x\}) \neq \phi \quad \forall \epsilon > 0$.

"every ϵ-neighborhood of x intersects A in something other than x".

Another way $\rightarrow$

Theorem: x is a limit point of A iff $x = \lim_{n\to\infty} a_n$ for some sequence $(a_n) \subseteq A \setminus \{x\}$

Proof (Outline) $\boxed{\Rightarrow}$ Let x be a limit point of A: take $\epsilon = 1/n$

Every $1/n$-neighborhood of x intersects $A \setminus \{x\}$, so pick $a_n \in \underbrace{V_{1/n}(x) \cap (A \setminus \{x\})}_{\neq \phi}$.

Verify $a_n \to x$.

<u>Defn</u> A point x is a <u>limit point</u> of a set A if $V_\epsilon(x) \cap (A \setminus \{x\}) \neq \emptyset$ $\forall \epsilon > 0$.

"every ϵ-neighborhood of x intersects A in something other than x."

<u>Another way</u> →

<u>Theorem</u>: x is a limit point of A iff

$$x = \lim_{n\to\infty} a_n \text{ for some sequence } (a_n) \subseteq A \setminus \{x\}$$

<u>Proof (Outline)</u> $\boxed{\Leftarrow}$ The definition of convergence tells us $\forall \epsilon > 0$, $a_n \in V_\epsilon(x)$ $\forall n \geq N$, so $a_N \in V_\epsilon(x)$ and $a_N \neq x$.

Also, $\bar{A}$ is the smallest closed set containing A

Defn For $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The closure of A is defined to be $\bar{A} = A \cup L$.

Defn Set A is closed if $A = \bar{A}$,
i.e., A contains all its limit points.

Theorem $A \subseteq \mathbb{R}$ is closed iff every Cauchy sequence in A has a limit also in A.

Proof Exercise!

Comment "Closed" under the operation of taking limits of sequences.

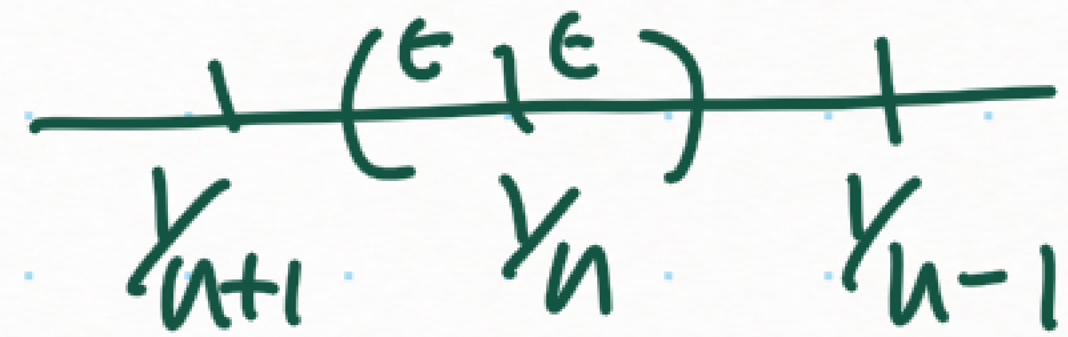
not a limit point of A

Examples

① $A = \{\frac{1}{n} : n \in \mathbb{N}\}$

Every point of A is isolated

Given $\frac{1}{n} \in A$, we need to find $\epsilon > 0$ s.t. $V_\epsilon(\frac{1}{n}) \cap A = \{1/n\}$



not a limit point of A

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① $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ Every point of A is isolated

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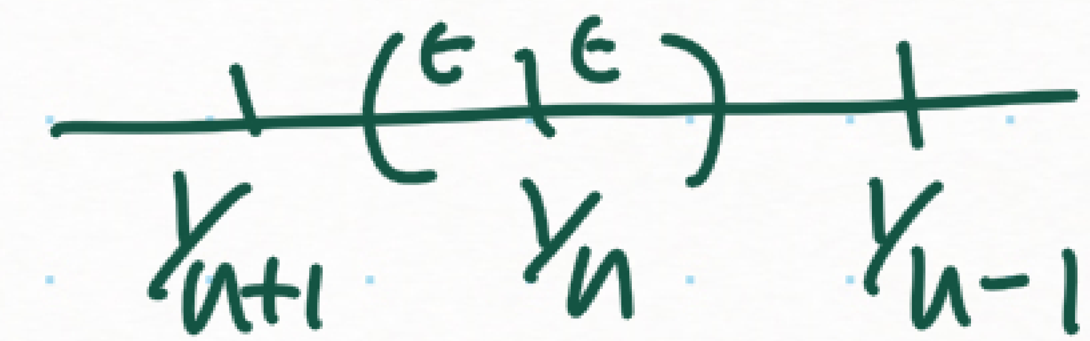


Pick $\epsilon = \frac{1}{n} - \frac{1}{n+1}$ (distance between $1/n$ & $\frac{1}{n+1}$)

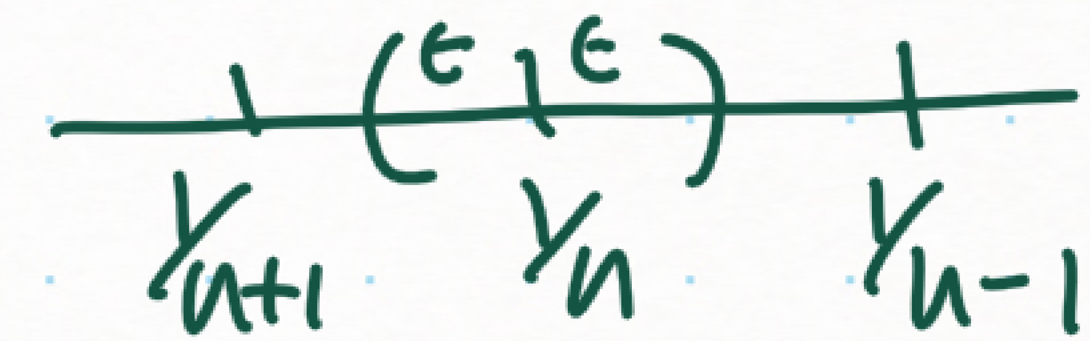
Any limit points of A?

not a limit point of A

Examples

① $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ Every point of A is isolated

Given $\frac{1}{n} \in A$, we need to find $\epsilon > 0$ s.t. $V_\epsilon(\frac{1}{n}) \cap A = \{1/n\}$



Pick $\epsilon = \frac{1}{n} - \frac{1}{n+1}$ (distance between $1/n$ & $\frac{1}{n+1}$)

Any limit points of A?

0 is the only limit point

Every $V_\epsilon(0)$ will intersect A (Why?)

$= (-\epsilon, \epsilon) \cap \{1/n : n \in \mathbb{N}\} \neq \emptyset$ $\exists n$ s.t. $1/n < \epsilon$

$\therefore \overline{A} = A \cup \{0\}$

② $[c,d]$ is a closed set

We want to prove every limit pt. of $[c,d]$ belongs to it.

If x is limit pt. of $[c,d]$ then

by Thm (x limit pt. of $A \Leftrightarrow x = \lim a_n$ for $(a_n) \subseteq A - \{x\}$),

we know $x = \lim x_n$ where $x_n \in [c,d] \setminus \{x\}$

Does $x \in [c,d]$?

since $c \leq x_n \leq d$, by Order limit Thm,

$c \leq \lim x_n \leq d$

i.e., $c \leq x \leq d$

i.e, $x \in [c,d]$ ✓

So, closure of $[c,d] = [c,d]$

③ Is the set $\mathbb{Q} \subseteq \mathbb{R}$ closed?

Let $x \in \mathbb{R}$ & $V_\epsilon(x) = (x-\epsilon, x+\epsilon)$ be any neighborhood of x

By Thm (Density of $\mathbb{Q}$ in $\mathbb{R}$), we know $\exists\, r \neq x$ s.t. $r \in (x-\epsilon, x+\epsilon) \cap \mathbb{Q}$

That is,

③ Is the set $\mathbb{Q} \subseteq \mathbb{R}$ closed?

Let $x \in \mathbb{R}$ & $V_\epsilon(x) = (x-\epsilon, x+\epsilon)$ be any neighborhood of x

By Thm (Density of $\mathbb{Q}$ in $\mathbb{R}$), we know $\exists r \neq x$ s.t. $r \in (x-\epsilon, x+\epsilon) \cap \mathbb{Q}$

That is, x is a limit point of $\mathbb{Q}$.

$$\boxed{\therefore \overline{\mathbb{Q}} = \mathbb{R}}$$

Theorem [Alternate form of Density of $\mathbb{Q}$ in $\mathbb{R}$]

For every $x \in \mathbb{R}$, $\exists$ seq. of rational numbers that converges to x.

(4) Is $\mathbb{R}$ closed?

(5) Is (a,b) closed? What is $\overline{(a,b)}$?

(6) Is $(a,b]$ closed? What is $\overline{(a,b]}$?

$\vdots$

→ Are there any sets that are both open and closed?

→ Are there any sets that are neither open nor closed?

complement of F

Theorem A set F is open $\Leftrightarrow$ F^c is closed

Since $(F^c)^c = F$, this means F is closed $\Leftrightarrow$ F^c is open.

complement of O

Theorem A set F is open $\Longleftrightarrow$ F^c is closed

Since $(F^c)^c = F$, this means F is closed $\Longleftrightarrow$ F^c is open.

Proof Let $F \subseteq \mathbb{R}$ be open

To show F^c is closed, we need to show it contains all its limit points.

Let x be a limit point of F^c

then $V_\epsilon(x) \cap (F^c \setminus \{x\}) \neq \phi \quad \forall \epsilon > 0$ — $(*)$

(i.e., every ϵ-neighborhood of x contains a pt. of F^c)

Claim $x \in F^c$

Pf. If $x \notin F^c$, i.e., $x \in F$ then $\exists \epsilon > 0$ s.t. $V_\epsilon(x) \subseteq F$ ($\because F$ open)

which is not possible by $(*)$

contradiction

complement of O

Theorem A set F is open $\Leftrightarrow$ F^c is closed

Since $(F^c)^c = F$, this means F is closed $\Leftrightarrow$ F^c is open.

Proof Let F^c be closed

To show F is open, ~~for~~ each $x \in F$ we must find $\epsilon > 0$
s.t. $V_\epsilon(x) \subseteq F$

Note that $x^{\in F}$ cannot be a limit point of F^c (since F^c is closed & contains all its limit points)

By negation of defn of limit pt., we have

$\exists \epsilon > 0$ s.t. $V_\epsilon(x) \cap F^c = \emptyset$

i.e, $V_\epsilon(x) \subseteq F$, as needed.

Using this characterization & properties of open sets, we have

<u>Theorem</u> ① Union of a finite collection of closed sets is closed

② Intersection of an arbitrary collection of closed sets is closed.

(<u>De Morgan's Laws</u>: $\left(\bigcup_{\lambda} A_{\lambda}\right)^{c} = \bigcap_{\lambda} A_{\lambda}^{c}$

$\left(\bigcap_{\lambda} A_{\lambda}\right)^{c} = \bigcup_{\lambda} A_{\lambda}^{c}$)