MATH 400

Real Analysis

Part #21

Heine-Borel Theorem

Let $K \subseteq \mathbb{R}$. The following statements are equivalent and characterize compact sets in $\mathbb{R}$:

① Every sequence in K has a subsequence that converges to a limit that is also in K.

② K is closed and bounded

③ Every open cover for K has a finite subcover.

Heine-Borel Theorem

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sequential compactness

② K is closed and bounded

③ Every open cover for K has a finite subcover.

compactness

Defn $K \subseteq \mathbb{R}$ is sequentially compact if every sequence in K has a subseq. that converges to a limit that is in K.

e.g. $[c,d]$

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If $(a_n) \subseteq [c,d]$ then Bolzano-Weierstrass tells us (since $c \leq a_n \leq d \ \forall n$) there is a convergent subseq. (a_{n_k})

Since $[c,d]$ is a closed set, the limit of $(a_{n_k}) \in [c,d]$

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Defn $A \subseteq \mathbb{R}$ is bounded if $\exists M > 0$ s.t. $|a| < M$ $\forall a \in A$

<u>Theorem</u> $K \subseteq \mathbb{R}$ is sequentially compact $\iff$ K is closed and bounded

<u>Proof</u> Let K be sequentially compact

Suppose K is not a bounded set.

Since K is not bounded, $\exists\, x_1 \in K$ with $|x_1| > 1$,

$\exists\, x_2 \in K$ with $|x_2| > 2$, $\cdots\cdots$, $\exists\, x_n \in K$ with $|x_n| > n$

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Since K is sequentially compact, $\exists$ convergent subseq. (x_{n_k})

By defn. of (x_n), we have $|x_{n_k}| > n_k$, i.e. (x_{n_k}) is unbdd.

so, (x_{n_k}) can not be convergent, contradiction.

$\therefore K$ must be bounded.

K is closed, i.e. K contains all its limit points.

Let $x = \lim x_n$ where $(x_n) \subseteq K$

Claim $x \in K$

with limit also in K.

Since K is seq. compact, (x_n) has a cgt. subseq. (x_{n_k})

Since $x_n \to x$, x_{n_k} must also have the same limit x.

By defn. of seq. compactness, $x \in K$.

$\boxed{\Leftarrow}$ HW exercise.

Nested Compact Set Property

If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots\cdots$

is a nested sequence of nonempty compact sets

then $\bigcap_{n=1}^{\infty} K_n$ is not empty

"compact sets" capture the essense of

"Closed intervals"

Defn Let $A \subseteq \mathbb{R}$

An open cover for A is collection of open sets $\{U_\alpha ; \alpha \in S\}$ s.t. $A \subseteq \bigcup_{\alpha \in S} U_\alpha$.

If $\{U_\alpha\}_{\alpha \in S}$ has a finite subset $\{U_\alpha\}_{\alpha \in F}$ (i.e., $F \subseteq S$ and $|F| < \infty$) which is still a cover of A, i.e., $A \subseteq \bigcup_{\alpha \in F} U_\alpha$, then $\{U_\alpha\}_{\alpha \in F}$ is called a finite subcover of A.

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Defn $K \subseteq \mathbb{R}$ is a compact set if every open cover of K contains a finite subcover of K.

Examples

Consider $(0,1)$.

For each $x \in (0,1)$, let $U_x = \left(\frac{x}{2}, 1\right)$

x

Then $\{U_x\}_{x \in (0,1)}$ is an open cover of $(0,1)$

Examples

Consider $(0,1)$.

For each $x \in (0,1)$, let $U_x = (\frac{x}{2}, 1)$

Then $\{U_x\}_{x \in (0,1)}$ is an open cover of $(0,1)$

But there is no finite subcover here.

Consider any finite collection $\{U_{x_1}, U_{x_2}, \dots, U_{x_k}\}$

Then for $x' = \min\{x_1, \dots, x_k\}$

any y s.t. $0 < y < \frac{x'}{2}$ is not in $\bigcup_{i=1}^{k} U_{x_i}$.

$\therefore (0,1)$ is not compact.

Consider $[0,1]$

The same open cover as before $\underline{\{U_x = (\frac{x}{2}, 1)\}_{x\in(0,1)}}$

covers every point in $[0,1]$ except 0 & 1.

So, for a fixed $\overset{=0.1}{\epsilon} > 0$, let $U_0 = (-\underset{0}{\epsilon}, \epsilon)$ & let $U_1 = (1-\epsilon, \underset{1}{1+\epsilon})$

Now, $\underline{\{U_0, U_1, U_x : x\in(0,1)\}}$ is an open cover of $[0,1]$

Does this open cover have a finite subcover?

Consider $[0,1]$

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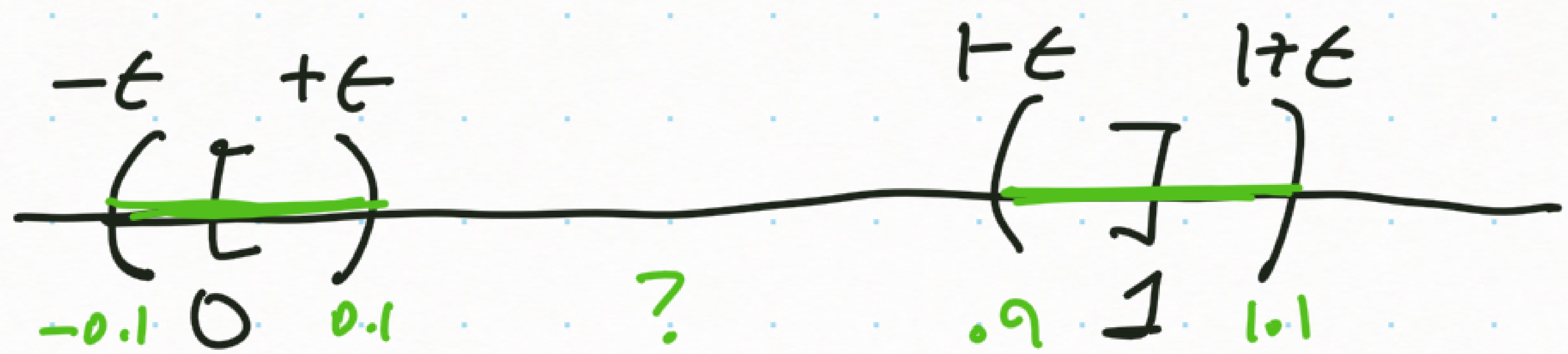

(Think of $\epsilon = 0.1$)

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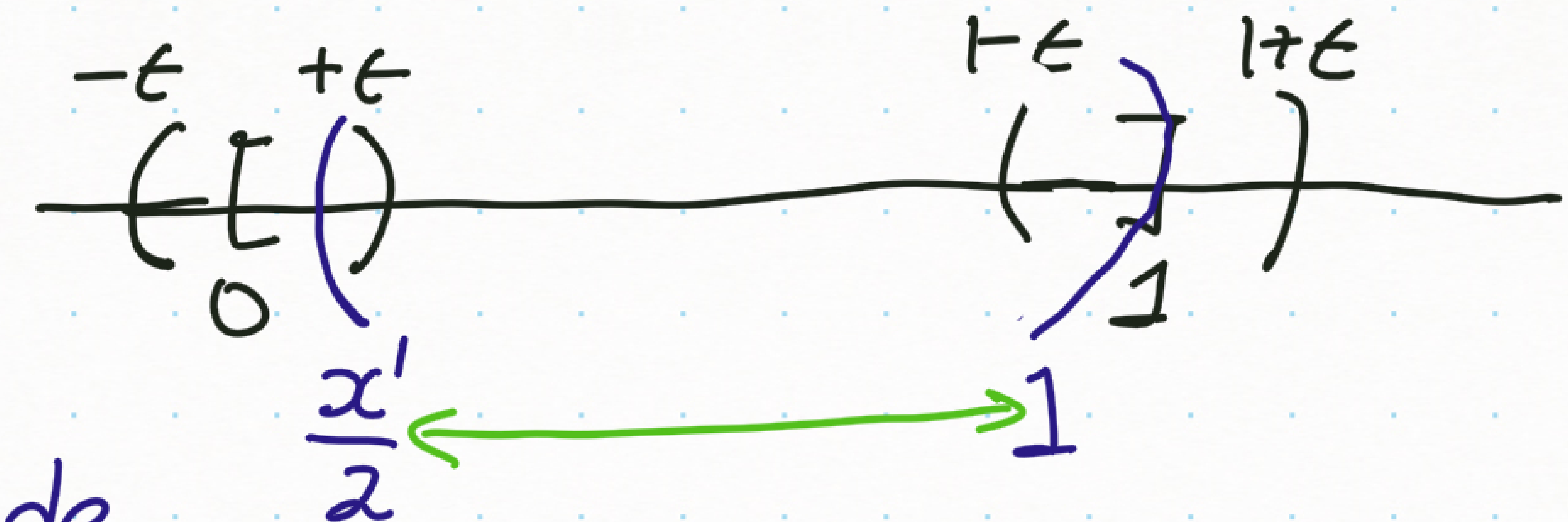

(Think of $\epsilon = 0.1$)

Also include

$(\frac{x'}{2}, 1)$ where $\frac{x'}{2} < \epsilon$ (which is possible to find).

Consider $(0, 4)$ and its open cover

$$\left\{\left(\frac{1}{k}, 4-\frac{1}{k}\right)\right\}_{k=1}^{\infty} = \left\{(1,3), \left(\frac{1}{2}, 4-\frac{1}{2}\right), \left(\frac{1}{3}, 4-\frac{1}{3}\right), \ldots\right\}$$

→ Check every $x \in (0,4)$ belongs to this cover

→ Check there is no finite subcover in this cover.

$\therefore (0,4)$ is not compact.

Theorem Let $K \subseteq \mathbb{R}$.

K is compact (every open cover has a finite subcover)

iff K is closed and bounded.

Proof $\boxed{\Rightarrow}$ We assume K is compact, that is every open cover of K has a finite subcover.

K is bounded

Let $I_n = (-n, n)$, then $\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$

& since $K \subseteq \mathbb{R}$, $K \subseteq \bigcup_{n=1}^{\infty} I_n$, an open cover of K.

$$K \subseteq \bigcup_{n=1}^{\infty} I_n \text{ , an open cover}$$

so, there is a finite subcover $\{I_{n_1}, I_{n_2}, \dots, I_{n_k}\}$.

Assume $n_1 < n_2 < \dots < n_k$

then $I_{n_1} = (-n_1, n_1) \subseteq I_{n_2} = (-n_2, n_2) \subseteq \dots \dots \subseteq I_{n_k}$

Hence $K \subseteq \bigcup_{i=1}^{k} I_{n_i} = I_{n_k} = (-n_k, n_k)$

i.e, every element of $K \in (-n_k, n_k)$

i.e, K is bounded.

K is closed Suppose K is not closed.

Then $\exists\, x \notin K$ and $(a_n) \subseteq K$ s.t. $\lim_{n\to\infty} a_n = x$ — ⊛

Let $U_n = (-\infty, x-\frac{1}{n}) \cup (x+\frac{1}{n}, \infty)$, open set

$\mathbb{R} - [x-\frac{1}{n}, x+\frac{1}{n}]$

and $\bigcup_{n=1}^{\infty} U_n = \mathbb{R} - \{x\}$.

Since $K \subseteq \mathbb{R} \sim \{x\}$, $K \subseteq \bigcup_{n=1}^{\infty} U_n$, an open cover of K.

By compactness of K, $\{U_n\}_{n=1}^{\infty}$ contains a finite subcover of K: $\{U_{n_1}, U_{n_2}, \dots, U_{n_k}\}$

Assume $n_1 < n_2 < \dots < n_k$, & thus $K \subseteq \bigcup_{\ell=1}^{k} U_{n_\ell} = U_{n_k}$

$K \subseteq U_{n_k}$ means $K \subseteq \left(-\infty, x - \frac{1}{n_k}\right) \cup \left(x + \frac{1}{n_k}, \infty\right)$

which implies $K \cap \left(x - \frac{1}{n_k}, x + \frac{1}{n_k}\right) = \emptyset$

Why is this a problem?

there are no elements of K within distance $\frac{1}{n_k}$ of x

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From $(*)$ we know $(a_n) \subseteq K$ and $\lim a_n = x$

If we let $\epsilon = \frac{1}{n_k} > 0$, then $(**)$ says

there is no a_n s.t. $|a_n - x| < \epsilon$

contradicting the definition of convergence for $\lim a_n = x$.

Math 400

Real Analysis

Part #22

What does it mean for a function to be continuous?

- Middle school $\rightarrow$ It looks like $f(x)=x^2$
- High school $\rightarrow$ You can draw it without picking up your pencil
- Pre-calculus $\rightarrow$ It does not have any holes or jumps.
- Calculus $\rightarrow$ If for each c, $\lim_{x \to c} f(x) = f(c)$

$\lim_{n \to \infty} a_n = a$

Recall (a_n) converges to a if

For all $\epsilon > 0$, $\exists N$ s.t. $|a_n - a| < \epsilon$ $\forall n > N$

How can we adapt this idea to define $\lim_{x \to c} f(x) = L$

we want $f(x)$ to be close to L when x is close to c

$|f(x) - L| < \epsilon$ — arbitrary

$|x - c| < \delta$ — chosen (like N)

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For all $\epsilon>0$, $\exists N$ s.t. $|a_n-a|<\epsilon \;\forall n>N$

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$|x-c|<\delta$ — chosen (like N)

<u>Defn</u> Let $f:A\to\mathbb{R}$ and let c be a limit pt. of A.
We say $\lim_{x\to c} f(x)=L$ if $\forall\epsilon>0, \exists\delta>0$ s.t.
$|f(x)-L|<\epsilon$ for every $x\in A$ which $0<|x-c|<\delta$

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Comments

① c need not be in A (domain of f).
As long as c is a limit point of A, we can pick points in A that approach it.

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③ $\boxed{|x - c| < \delta}$ means $x \in (c - \delta, c + \delta) = V_\delta(c)$, δ-neighborhood of c.
$0 < |x - c|$ forces $x \neq c$, that is we don't care what happens at $x = c$

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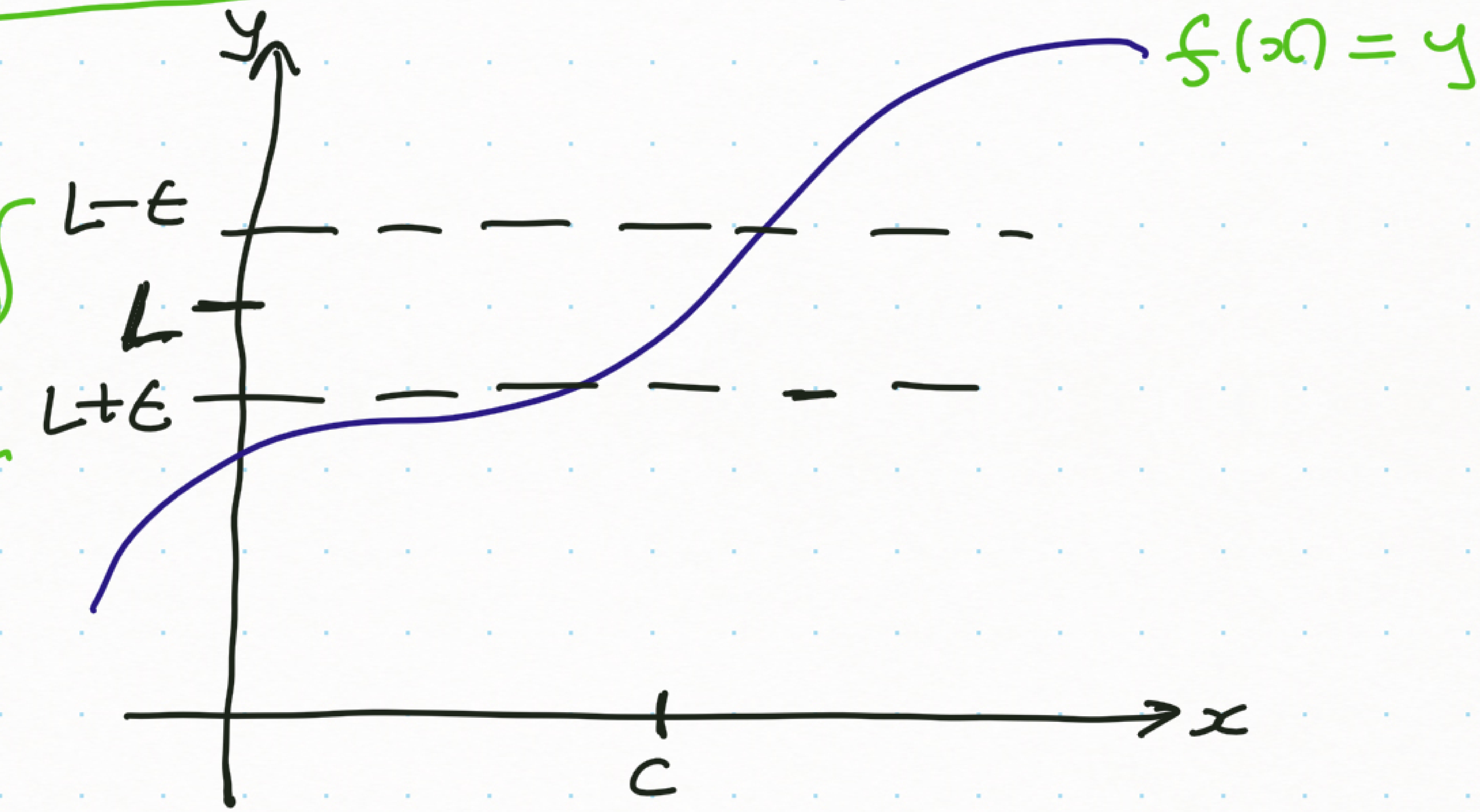
4

Given $\epsilon > 0$

ϵ-neighborhood of L



Find $\delta > 0$

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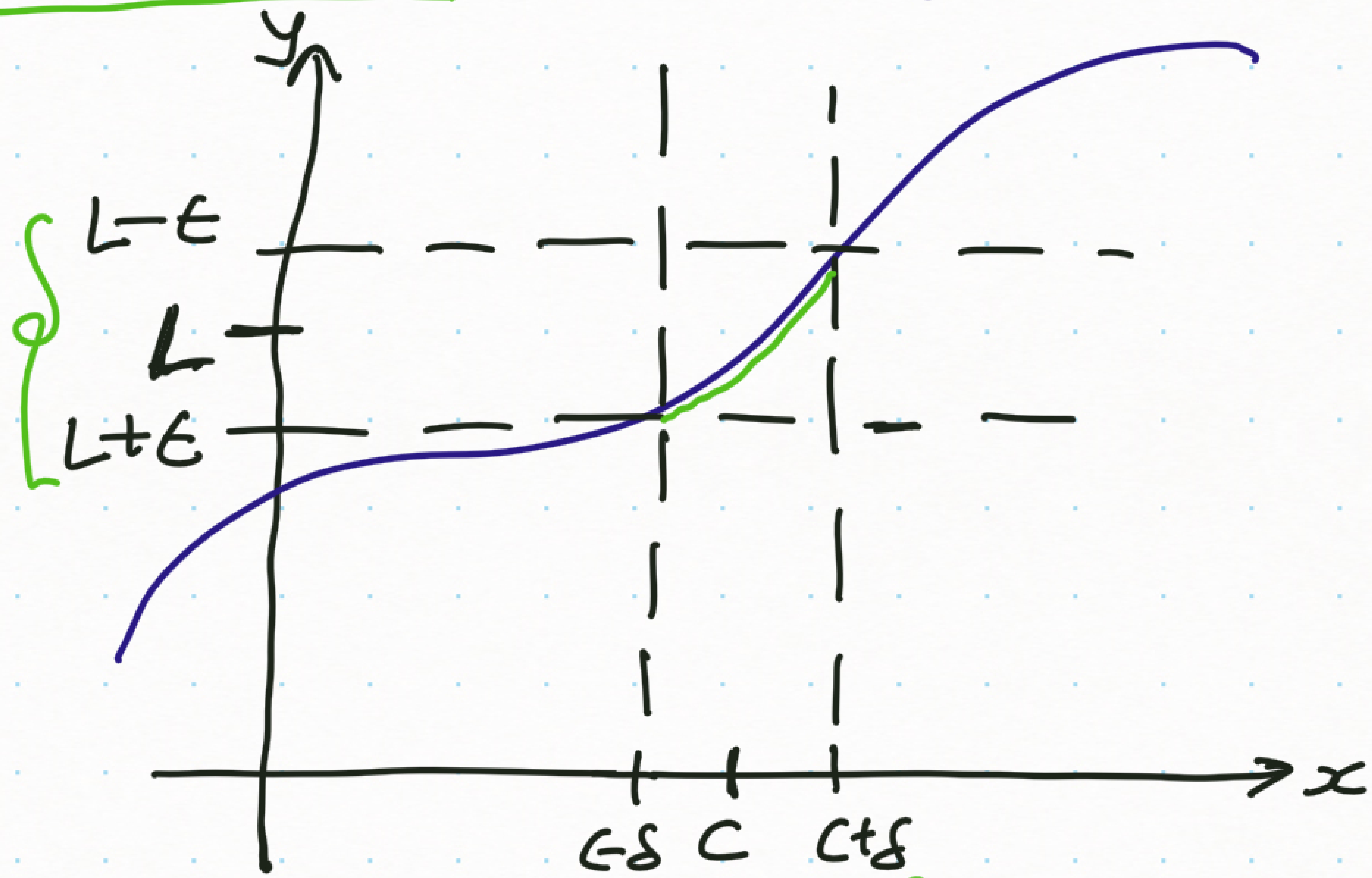
④

Given $\epsilon > 0$



Given these horizontal lines
Find vertical lines
s.t. $f(x)$ is trapped within the box

$\exists \delta > 0$: $f(x) \in \epsilon$-neighborhood of L
for $x \in \delta$-neighborhood of c

(5) To show $\lim_{x \to c} f(x)$ is not L

apply negation of definition.

$\exists\, \epsilon > 0$ s.t. for all $\delta > 0$, $\exists\, x \in A$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$.

There is an $\epsilon > 0$ s.t. we can not "trap" $f(x)$ in the "box" no matter which $\delta > 0$ we choose.

defined by ϵ-neighborhood of L and δ-neighborhood of c

Examples

① $f(x) = 5x+2$. $\lim_{x \to 3} f(x) = 17$

scratch work convert $|f(x)-L| < \epsilon$ into $|x-c| < \delta$

i.e., $|5x+2-17| < \epsilon$ i.e, $|5x-15| < \epsilon$, i.e., $5|x-3| < \epsilon$

i.e., $|x-3| < \epsilon/5$ so choose $\delta = \epsilon/5$

Soln Let $\epsilon > 0$, set $\delta = \frac{\epsilon}{5} > 0$ then for any x: $0 < |x-3| < \delta$

$$\begin{aligned} |f(x)-L| = |5x+2-17| = |5x-15| &= 5|x-3| \\ &< 5\delta \\ &= 5\frac{\epsilon}{5} \quad \text{(by choice of } \delta) \\ &= \epsilon \end{aligned}$$

② $f : \mathbb{R} - \{2\} \to \mathbb{R}$ with $f(x) = \dfrac{3x^2 - 12}{x-2}$. $\lim_{x \to 2} f(x) = 12$.

Scratch work: Convert $|f(x) - L| < \epsilon$ to $|x - c| < \delta$

$|f(x) - L| < \epsilon$ i.e., $\left| \dfrac{3x^2-12}{x-2} - 12 \right| < \epsilon$, i.e.,

$\left| \dfrac{3(x-2)(x+2)}{x-2} - 12 \right| < \epsilon$, i.e., $|3(x+2) - 12| < \epsilon$, i.e.,

$3|(x+2) - 4| < \epsilon$, i.e., $3|x-2| < \epsilon$, i.e., $|x-2| < \epsilon/3$

Soln. Let $\epsilon > 0$, set $\delta = \dfrac{\epsilon}{3} > 0$,

Then for any x : $0 < |x-2| < \delta$

$|f(x) - L| = \left| \dfrac{3x^2-12}{x-2} - 12 \right| = \cdots\cdots = 3|x-2|$

$< 3\delta$

$= 3\dfrac{\epsilon}{3} = \epsilon.$

③ $g(x) = x^2$. $\lim_{x \to 2} g(x) = 4$

scratch work $|x^2 - 4| < \epsilon$ $\rightsquigarrow$ $|x-2| < \delta$

$|x^2 - 4| = |(x+2)(x-2)| = |x+2|\,|x-2| < \epsilon$ compare

i.e, $|x-2| < \dfrac{\epsilon}{|x+2|}$ $\Leftarrow$ ~~Let $\delta = \frac{\epsilon}{|x+2|}$~~ not allowed need a number

we need to choose δ s.t.

when we do $0 < |x-2| < \delta \Rightarrow |x^2 - 4| = |x+2|\,|x-2|$

$< K\delta$

& we can pick $\delta = \frac{\epsilon}{K}$

So, we need K s.t. $|x+2| < K$ when $|x-2| < \delta$

i.e, $x < 2 + \delta$

③ $g(x) = x^2$. $\lim_{x \to 2} g(x) = 4$

$\Longleftarrow$

scratch work $\boxed{|x^2 - 4| < \epsilon}$ $\rightsquigarrow$ $\boxed{|x-2| < \delta}$

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i.e, $|x-2| < \dfrac{\delta}{|x+2|}$ ← not allowed need a number

We need to choose δ s.t.

when we do $0 < |x-2| < \delta \Rightarrow |x^2 - 4| = |x+2|\,|x-2|$
$< K\delta$

& we can pick $\boxed{\delta = \frac{\epsilon}{K}}$

So, we need K s.t. $|x+2| < K$ when $x < 2+\delta$

If we ensure $\underline{\delta \le 1}$ then $|x+2| \le |\underline{(2+1)} + 2| = 5$

$|x-2| < \delta$ means $\underline{x < 2+\delta}$
$\underline{\le 2+1}$

$\therefore$ choose $\boxed{\delta = \min\{1, \epsilon/5\}}$

Theorem (Sequential Criterion for Functional Limits)

Given $f: A \to \mathbb{R}$ and limit point c of A, the following are equivalent:

(i) $\boxed{\lim_{x \to c} f(x) = L.}$

(ii) For all $(x_n) \subseteq A$ with $x_n \neq c$, & $x_n \to c$, $\boxed{f(x_n) \to L.}$

$\underbrace{\text{sequence}}$

$\underbrace{\text{defn of convergence of sequence}}$

Use any Theorems or properties of convergent seq

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(ii) $\boxed{\text{For all } (x_n) \subseteq A \text{ with } x_n \neq c, \ \& \ x_n \to c, \quad f(x_n) \to L.}$

Cor Let $f, g : A \to \mathbb{R}$ with $\lim_{x\to c} f(x) = L$ & $\lim_{x \to c} g(x) = M$

(i) $\lim_{x \to c} kf(x) = kL$ for all $k \in \mathbb{R}$

(ii) $\lim_{x\to c} (f(x) + g(x)) = L + M$

(iii) $\lim_{x\to c} (f(x)g(x)) = LM$

(iv) $\lim_{x \to c} f(x)/g(x) = L/M$ provided $M \neq 0$