MATH 400

Real Analysis

Part #23

Theorem (Sequential Criterion for Functional Limits)

Given $f: A \to \mathbb{R}$ and limit point c of A, the following are equivalent:

(i) $\lim_{x \to c} f(x) = L$.

(ii) For all $(x_n) \subseteq A$ with $x_n \neq c$, & $x_n \to c$, $f(x_n) \to L$.

Cor Let $f, g : A \to \mathbb{R}$ with $\lim_{x \to c} f(x) = L$ & $\lim_{x \to c} g(x) = M$

(i) $\lim_{x \to c} kf(x) = kL$ for all $k \in \mathbb{R}$

(ii) $\lim_{x \to c} (f(x) + g(x)) = L + M$

(iii) $\lim_{x \to c} (f(x)g(x)) = LM$

(iv) $\lim_{x \to c} f(x)/g(x) = L/M$ provided $M \neq 0$

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Cor (Divergence Criterion)

Given $f: A \to \mathbb{R}$ and limit point c of A.

If $\exists$ $(x_n), (y_n) \subseteq A$ with $x_n \neq c$ and $y_n \neq c$, and $\lim x_n = c = \lim y_n$ but $\lim f(x_n) \neq \lim f(y_n)$

Then $\lim_{x \to c} f(x)$ does not exist.

Theorem (Sequential Criterion for Functional Limits)

Given $f: A \to \mathbb{R}$ and limit point c of A, the following are equivalent:

(i) $\lim_{x \to c} f(x) = L$.

(ii) For all $(x_n) \subseteq A$ with $x_n \neq c$, & $x_n \to c$, $f(x_n) \to L$.

Proof

$\boxed{\Rightarrow}$ Let $\epsilon > 0$.

$\lim_{x\to c} f(x) = L$ means $\exists \delta > 0$ s.t. $|f(x) - L| < \epsilon$ for $0 < |x - c| < \delta$ — (*)

Let $(x_n) \subseteq A$ be an arbitrary seq convergent to c

Since $x_n \to c$, $\exists N$ s.t. $|x_n - c| < \delta$ $\forall n > N$ (using $\delta > 0$ as ϵ in defn. of $x_n \to c$)

(& since $x_n \neq c$, $0 < |x_n - c|$ is automatic)

$\therefore |f(x_n) - L| < \epsilon$ from (*)

$\boxed{\Leftarrow}$ Assume $f(x_n) \to L$ for every $(x_n) \subseteq A - \{c\}$ with $x_n \to c$

Assume $\lim_{x \to c} f(x) \neq L$, i.e.,

$\exists\, \epsilon > 0$ s.t. $\underline{\forall\, \delta > 0}$ $\exists\, x \in A$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$

Taking this $\epsilon > 0$ and setting $\delta_n = \frac{1}{n}$, we get the existence of $x_n \in A$ with $0 < \underline{|x_n - c| < \frac{1}{n}}$ and $|f(x_n) - L| \geq \epsilon$

what does this tell us about (x_n)?

$\boxed{\Leftarrow}$ Assume $f(x_n) \to L$ for every $(x_n) \subseteq A \setminus \{c\}$ with $x_n \to c$

Assume $\lim_{x \to c} f(x) \neq L$, i.e.,

$\exists\, \epsilon > 0$ s.t. $\forall\, \delta > 0$ $\exists\, x \in A$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$

Taking this $\epsilon > 0$ and setting $\delta_n = \frac{1}{n}$, we get the existence of $x_n \in A$ with $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \epsilon$

Since $|x_n - c| < \frac{1}{n}$ $\forall n$, we have found a sequence $x_n \in A \setminus \{c\}$ with $x_n \to c$.

But $|f(x_n) - L| \geq \epsilon$ $\forall n$, so $f(x_n) \not\to L$.

Contradiction.

f is continuous at a point c if $\lim_{x \to c} f(x) = f(c)$

more precisely →

Defn $f : A \to \mathbb{R}$ is continuous at $c \in A$

if $\forall \epsilon > 0 \quad \exists \delta > 0$ s.t. $|f(x) - f(c)| < \epsilon \quad \forall x \in A$ with $|x - c| < \delta$

If f is continuous at every point in its domain, then f is called continuous.

Note the differences with "limit" definition

- $c \in A$
- $x = c$ allowed

Theorem [Summary of Definitions of Continuity]

Let $f : A \to \mathbb{R}$ and $c \in A$. Then the following are equivalent:

① f is continuous at c

② $\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $|f(x) - f(c)| < \epsilon \ \forall x \in A$ with $|x - c| < \delta$

③ $\forall V_\epsilon(f(c)) \ \exists V_\delta(c)$ s.t. $f(x) \in V_\epsilon(f(c)) \ \forall x \in V_\delta(c) \cap A$

④ $\lim\limits_{x \to c} f(x) = f(c)$ if c is a limit pt. of A

⑤ $\forall (a_n) \subseteq A$ with $\lim a_n = c$, we have $\lim f(a_n) = f(c)$

Sequential characterization of continuity gives us:

Cor [Discontinuity Criterion]

① If $\exists\ (a_n) \subseteq A$ with $a_n \to c$ but $f(a_n) \not\to f(c)$ then f is discontinuous at c.

② If $\exists\ (a_n), (b_n) \subseteq A$ with $a_n \to c$ and $b_n \to c$ but $\lim f(a_n) \neq \lim f(b_n)$, then f is discontinuous at c.

Sequential characterization of continuity and algebra of functional limits gives us:

Theorem [Algebra of Continuous functions]

Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be continuous at $c \in A$, then

① $k f(x)$ is continuous at c, for any $k \in \mathbb{R}$

② $f(x) + g(x)$ is continuous at c

③ $f(x) \cdot g(x)$ is continuous at c

④ $\dfrac{f(x)}{g(x)}$ is continuous at c, provided $g(x) \neq 0 \ \forall x \in A$.

Another application of sequential characterization of continuity

Theorem [f, g continuous $\Rightarrow f \circ g$ continuous]

Let $g: A \to B$ and $f: B \to \mathbb{R}$.

If g is continuous at $c \in A$ and f is continuous at $g(c) \in B$

then $f \circ g: A \to \mathbb{R}$ is continuous at c.

Algebra and composition of continuous functions allow us to build complicated continuous functions from simpler ones.

- $f(x) = x$ is continuous [Verify the definition]

Algebra and composition of continuous functions allow us to build complicated continuous functions from simpler ones.

- $f(x) = x$ is continuous [Verify the definition]
- $f(x) = x^n$ is continuous for each $n \in \mathbb{N}$
- $f(x) = a_n x^n$ ——— " ——— $n \in \mathbb{N}$ & $a_n \in \mathbb{R}$
- $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ is continuous
- Rational function: $\dfrac{a_0 + a_1 x + \cdots + a_k x^k}{b_0 + b_1 x + \cdots + b_m x^m}$ is continuous wherever it is defined.

Algebra and composition of continuous functions allow us to build complicated continuous functions from simpler ones.

- $f(x) = \sqrt{x}$ defined on $A = \{x \in \mathbb{R} : x \geq 0\}$ is continuous

  [• See Ex. 2.3.1 from HW#3 for a sequential proof
  • Give a direct ϵ-δ proof see pg. 125 of the textbook]

- $f(x) = \sqrt{a_0 + a_1 x + \dots + a_k x^k}$ is continuous wherever its defined since $f(x) = g(h(x))$ where $g(x) = \sqrt{x}$
  $h(x) = a_0 + a_1 x + \dots + a_k x^k$
  are both continuous.

- $\lim_{x \to 0} \sin(1/x)$ does not exist

Let
$x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{2n\pi + \pi/2}$

then $x_n \to 0$ and $y_n \to 0$

But, $\sin(1/x_n) = 0$ & $\sin(1/y_n) = 1$

so, $\lim(\sin(1/x_n)) \neq \lim(\sin(1/y_n))$

- $\lim_{x\to 0} \sin(1/x)$ does not exist

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so, $\lim(\sin(1/x_n)) \neq \lim(\sin(1/y_n))$

But $g(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous at $c = 0$.

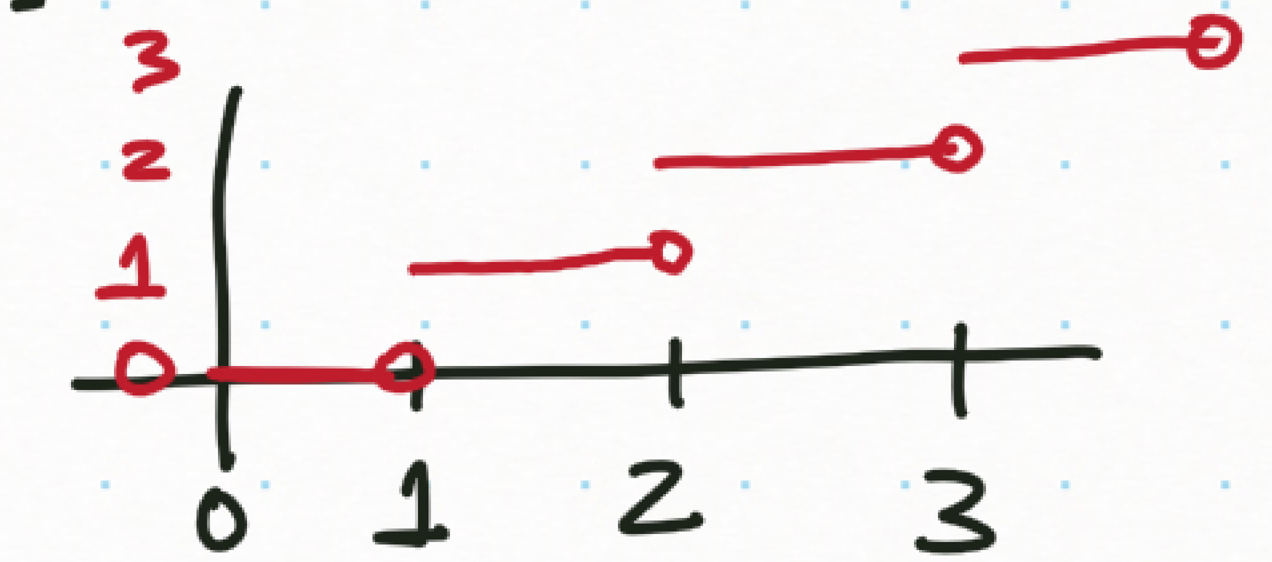
Note

$$|g(x) - g(0)| = |x\sin(1/x) - 0| = |x\sin(1/x)| \leq |x|\,|\sin(1/x)| \leq |x|$$

Given $\epsilon > 0$, set $\delta = \epsilon$ so $|x - 0| = |x| < \delta$

$$\Rightarrow |g(x) - g(0)| \leq |x| < \epsilon$$

- $h(x) = [[x]]$, largest integer at most x



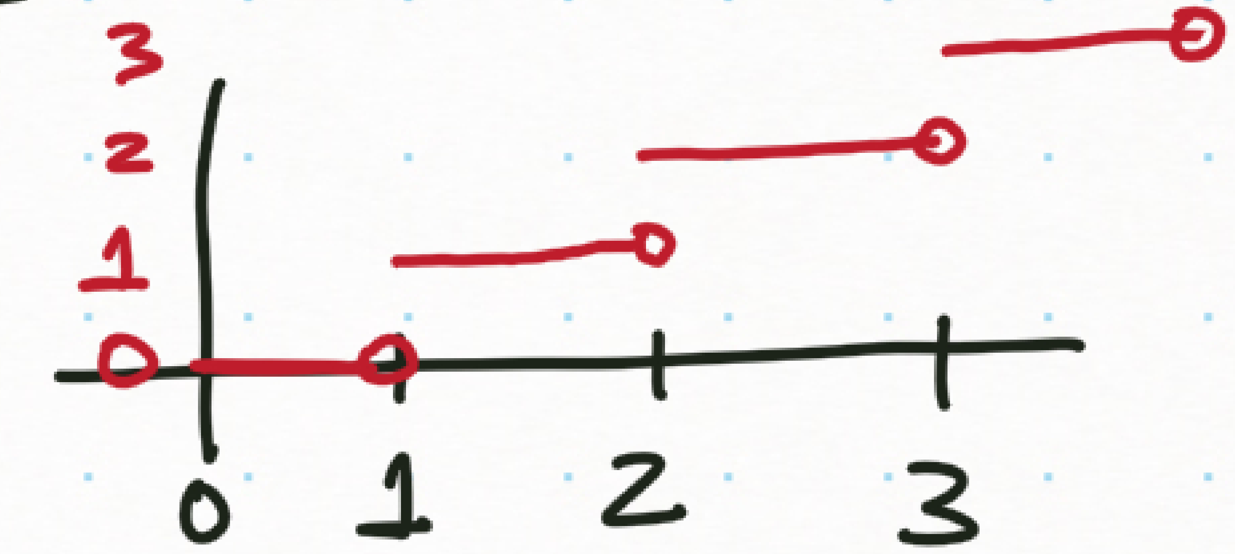
h is discontinuous at each $m \in \mathbb{Z}$

Define $x_n = m - \frac{1}{n}$. Then $x_n \to m$ as $n \to \infty$

But $h(x_n) = m-1 \to m-1$

$\neq m = h(m)$

• $h(x) = [[x]]$, largest integer at most x



h is discontinuous at each $m \in \mathbb{Z}$

Define $x_n = m - \frac{1}{n}$. Then $x_n \to m$

But $h(x_n) = m-1 \to m-1$
$\neq m = h(m)$

h is continuous at each $c \notin \mathbb{Z}$

Let $c \notin \mathbb{Z}$ Then $c \in (m, m+1)$ for some $m \in \mathbb{Z}$

Given $\epsilon > 0$, we need $\delta > 0$ s.t. $x \in V_\delta(c) \Rightarrow h(x) \in V_\epsilon(h(c))$

Let $\underline{\delta = \min\{c-m, (m+1)-c\}}$ then $h(x) = h(c)$ $\forall x \in (c-\delta, c+\delta)$
& hence $h(x) \in V_\epsilon(h(c))$.

Does there exist a function that is discontinuous everywhere (continuous nowhere)?

Does there exist a function that is discontinuous everywhere (continuous nowhere)?

Dirichlet's Function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- Recall $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{I}$ is dense in $\mathbb{R}$

- Show f is discontinuous at each $c \in \mathbb{R}$

Idea: Find $(a_n) \subseteq \mathbb{Q}$ s.t. $a_n \to c$ ✓

Find $(b_n) \subseteq \mathbb{I}$ s.t. $b_n \to c$ ✓

Then $f(a_n) = 1 \ \forall n$ & $f(b_n) = 0 \ \forall n$

so, $\lim f(a_n) \neq \lim f(b_n)$

Does there exist a function that is continuous at exactly one point and discontinuous everywhere else?

Does there exist a function that is continuous at exactly one point and discontinuous everywhere else?

Modified Dirichlet's Function

$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- Show g is discontinuous at each $c \neq 0$ (construct sequences)
- Show g is continuous at $c = 0$

Can a function be continuous at every irrational and discontinuous at every rational number?

Can a function be continuous at every irrational and discontinuous at every rational number?

Thomae's Function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- Show t is discontinuous at each $c \in \mathbb{Q}$.

[Since $t(x) > 0$ for $x \in \mathbb{Q}$ while $t(x) = 0$ for $x \notin \mathbb{Q}$]

- Show t is continuous at each $c \notin \mathbb{Q}$.

But this idea doesn't work here!!

Math 400

Real Analysis

Part #24

Given $f : A \to \mathbb{R}$

For any $B \subseteq A$, $f(B) = \{f(x) : x \in B\}$
range of f over B

Which properties of B are preserved in $f(B)$ when f is a continuous function?

→ Open?

→ Closed?

⋮

If B is open & f is continuous then $f(B)$ is open?

Consider $f(x) = x^2$, $B = (-1, 1)$

$f(B) = [0, 1)$

If B is open & f is continuous then $f(B)$ is open?

Consider $f(x) = x^2$, $B = (-1, 1)$

$f(B) = [0, 1)$ not open

If B is closed & f is continuous then $f(B)$ is closed?

Consider $g(x) = \dfrac{1}{1+x^2}$, $B = [0, \infty)$

$g(B) =$

If B is open & f is continuous then $f(B)$ is open?

Consider $f(x) = x^2$, $B = (-1, 1)$ open

$f(B) = [0, 1)$ not open

If B is closed & f is continuous then $f(B)$ is closed?

Consider $g(x) = \dfrac{1}{1+x^2}$, $B = [0, \infty)$ closed

$g(B) = (0, 1]$ not closed

What if B is closed and bounded?

Theorem [Preservation of Compact Sets]

Let $f: A \to \mathbb{R}$ be continuous on A.
If $K \subseteq A$ is compact then $f(K)$ is also compact

Proof we will show $f(K)$ is sequentially compact.

Let $(y_n) \subseteq f(K)$ be an arbitrary seq.
i.e., for each $n \in \mathbb{N}$, $\exists\, x_n \in K$ s.t. $f(x_n) = y_n$

$\therefore$ (x_n) is a seq in K

Since K is compact, $\exists$ subseq (x_{n_k}) with $\lim_{k\to\infty} x_{n_k} = x \in K$

Since f is continuous, we have $f(x_{n_k}) \to f(x) \in f(K)$

i.e., $\lim_{k\to\infty} y_{n_k} = f(x) \in f(K)$.

$\therefore$

Extreme Value Theorem

If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains its maximum and minimum value in K.

In other words, $\exists\, x_0, x_1 \in K$ s.t. $f(x_0) \leq f(x) \leq f(x_1)$
$\forall x \in K$.

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Proof Since $f(K)$ is compact (i.e., closed & bounded),

$\alpha = \sup f(K)$ exists, and $\beta = \inf f(K)$ exists

By Ex 3.3.1, $\alpha \in f(K)$ and $\beta \in f(K)$.

$\therefore$ $\exists\, x_1 \in K$ s.t. $\alpha = f(x_1)$ and $\exists\, x_0 \in K$ s.t. $\beta = f(x_0)$

So, $f(x_0) = \beta \leq f(x) \leq \alpha = f(x_1)$ $\forall x \in K$.

Some examples

① $f:[-2,1]\longrightarrow \mathbb{R}$ as $f(x)=x^2$ continuous fn. on closed int.

$f([-2,1])$ has supremum $=4$ and infimum $=0$

And $\exists\, -2\in[-2,1]$ s.t. $f(-2)=4$, the sup is achieved

$\exists\, 0\in[-2,1]$ s.t. $f(0)=0$, the inf is achieved

② $f:(-2,1]\longrightarrow \mathbb{R}$ as $f(x)=x^2$ cont. fn. on non-closed set

Again the supremum $=4$ and infimum $=0$

But there is no $c\in(-2,1]$ with $f(c)=4$.

③ $f:[0,4]\longrightarrow\mathbb{R}$ as $f(x)=\begin{cases}2x & \text{if } x\in[0,2)\\ 2 & \text{if } x\in[2,4]\end{cases}$ not cont.

Here sup is 4 but $\nexists\, c\in[0,4]$ with $f(c)=4$.

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Real Analysis

Part #25

Bolzano's Theorem

Let f be continuous on $[a,b]$.
If $f(a)$ and $f(b)$ have different signs (positive or negative)
then $\exists\, c \in [a,b]$ s.t. $f(c)=0$ (solution of $f(x)=0$)

Proof

It is enough to prove:

$$\boxed{f(a)<0 \text{ and } f(b)>0 \Rightarrow f(c)=0 \text{ for some } c\in[a,b]}$$

[When $f(a)>0$ and $f(b)<0$
we can apply the above statement to $-f(x)$
to get $-f(a)<0$ & $-f(b)>0 \Rightarrow -f(c)=0$ for some $c\in[a,b]$

We will give a sequence $[a_i, b_i]$ of nested closed intervals s.t. $f(a_i) < 0$ & $f(b_i) > 0$ $\forall i$

Then c will be the unique pt. in $\bigcap_{i=0}^{\infty} [a_i, b_i]$.

bisection procedure

Let $[a_0, b_0] = [a, b]$ where $f(a_0) < 0$ and $f(b_0) > 0$

Divide $[a_0, b_0]$ into two intervals by its midpt. $x_0 = \frac{a_0 + b_0}{2}$

$$\left.\begin{array}{l} \text{If } f(x_0) > 0 \text{ then } [a_1, b_1] = [a_0, x_0] \\ \text{If } f(x_0) < 0 \text{ then } [a_1, b_1] = [x_0, b_0] \end{array}\right\} \begin{array}{l} \text{so, } f(a_1) < 0 \\ \text{and } f(b_1) > 0 \end{array}$$

If $f(x_0) = 0$ then we have found $c = x_0$ s.t. $f(c) = 0$.

We will give a sequence $[a_i, b_i]$ of nested closed intervals s.t. $f(a_i) < 0$ & $f(b_i) > 0$ $\forall i$

Then c will be the unique pt. in $\bigcap_{i=0}^{\infty} [a_i, b_i]$.

Let $[a_0, b_0] = [a, b]$ where $f(a_0) < 0$ and $f(b_0) > 0$

Divide $[a_0, b_0]$ into two intervals by its midpt. $x_0 = \frac{a_0 + b_0}{2}$

$$\left.\begin{array}{l} \text{If } f(x_0) > 0 \text{ then } [a_1, b_1] = [a_0, x_0] \\ \text{If } f(x_0) < 0 \text{ then } [a_1, b_1] = [x_0, b_0] \end{array}\right\} \begin{array}{l} \text{so, } f(a_1) < 0 \\ \text{and } f(b_1) > 0 \end{array}$$

If $f(x_0) = 0$ then we have found $c = x_0$ s.t. $f(c) = 0$.

For $n \geq 0$, divide $[a_n, b_n]$ (where $f(a_n) < 0$ & $f(b_n) > 0$) into two intervals by its midpoint $x_n = \frac{a_n + b_n}{2}$.

$$\left.\begin{array}{l} \text{If } f(x_n) > 0 \text{ then } [a_{n+1}, b_{n+1}] = [a_n, x_n] \\ \text{If } f(x_n) < 0 \text{ then } [a_{n+1}, b_{n+1}] = [x_n, b_n] \end{array}\right\} \begin{array}{l} \text{so, } f(a_{n+1}) < 0 \\ \text{and } f(b_{n+1}) > 0 \end{array}$$

If $f(x_n) = 0$ then we have found $c = x_n$ s.t. $f(c) = 0$.

For $n \geq 0$, we have $[a_n, b_n]$ with $f(a_n) < 0$ and $f(b_n) > 0$

Divide $[a_n, b_n]$ into two intervals by its midpoint $x_n = \frac{a_n + b_n}{2}$

$$\left.\begin{array}{l} \text{If } f(x_n) > 0 \text{ then } [a_{n+1}, b_{n+1}] = [a_n, x_n] \\ \text{If } f(x_n) < 0 \text{ then } [a_{n+1}, b_{n+1}] = [x_n, b_n] \end{array}\right\} \begin{array}{l} \text{so } f(a_{n+1}) < 0 \\ \& \ f(b_{n+1}) > 0 \end{array}$$

If $f(x_n) = 0$ then we have found $c = x_n$ s.t. $f(c) = 0$.

If $f(x_n) \neq 0$ $\forall n$ then we have $[a, b] \supseteq [a_1, b_1] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$ an infinite sequence of nested closed intervals.

By Nested Interval Property, $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$, so $\exists\, c \in [a_n, b_n]$ $\forall n$

For $n \geq 0$, we have $[a_n, b_n]$ with $f(a_n) < 0$
and $f(b_n) > 0$

Divide $[a_n, b_n]$ into two intervals by its midpoint $x_n = \frac{a_n + b_n}{2}$

$$\left.\begin{array}{l} \text{If } f(x_n) > 0 \text{ then } [a_{n+1}, b_{n+1}] = [a_n, x_n] \\ \text{If } f(x_n) < 0 \text{ then } [a_{n+1}, b_{n+1}] = [x_n, b_n] \end{array}\right\} \begin{array}{l} \text{so } f(a_{n+1}) < 0 \\ \& \ f(b_{n+1}) > 0 \end{array}$$

If $f(x_n) = 0$ then we have found $c = x_n$ s.t. $f(c) = 0$.

If $f(x_n) \neq 0$ $\forall n$ then we have $[a, b] \supseteq [a_1, b_1] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$
an infinite sequence of nested closed intervals.

By Nested Interval Property, $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \phi$, so $\exists\, c \in [a_n, b_n]$ $\forall n$

we have sequences (a_n) & (b_n) with $f(a_n) < 0$ & $f(b_n) > 0$ $\forall n$

And, $b_n - a_n \to 0$. $\therefore$ $\lim a_n = c = \lim b_n$

Since f is continuous, $\lim f(a_n) = f(c) = \lim f(b_n)$.

$$\left.\begin{array}{l} \text{As } f(a_n) < 0, \ \lim f(a_n) \leq 0 \ \& \text{ hence } f(c) \leq 0 \\ \text{As } f(b_n) > 0, \ \lim f(b_n) \geq 0 \ \& \text{ hence } f(c) \geq 0 \end{array}\right\} \Rightarrow f(c) = 0.$$

Intermediate Value Theorem

Let $f:[a,b]\to\mathbb{R}$ be continuous.
If $L\in\mathbb{R}$ s.t. $f(a)\leq L\leq f(b)$ or $f(a)\geq L\geq f(b)$
then $\exists\, c\in[a,b]$ s.t. $f(c)=L$.

"If $f:[a,b]\to\mathbb{R}$ is continuous then f takes on all values between $f(a)$ and $f(b)$"

Its enough to prove

$$\boxed{f(a)\leq L\leq f(b) \Rightarrow \exists\, c\in[a,b] \text{ s.t. } f(c)=L}$$

When $f(a)\geq L\geq f(b)$, we can apply the above statement to $-f$ since $f(a)\geq L\geq f(b) \Rightarrow -f(a)\leq -L\leq -f(b)$ & hence $\exists\, c$ s.t. $-f(c)=-L$

Intermediate Value Theorem

Let $f: [a,b] \to \mathbb{R}$ be continuous.
If $L \in \mathbb{R}$ s.t. $f(a) \leq L \leq f(b)$ or $f(a) \geq L \geq f(b)$
then $\exists\, c \in [a,b]$ s.t. $f(c) = L$.

Proof Assume $f(a) \leq L \leq f(b)$.
If $f(a) = L$ or $f(b) = L$ then we are done.
Else, $f(a) < L < f(b)$.
Define $g(x) = f(x) - L$. Then $g(a) = f(a) - L < 0$
and $g(b) = f(b) - L > 0$
Note g is continuous.
So, we can apply Bolzano's thm, to get
$\exists\, c \in [a,b]$ s.t. $g(c) = 0$, i.e, $f(c) - L = 0$, i.e., $f(c) = L$.

note: Bolzano's thm is special case of IVT for $L = 0$, but....

Example

$e^x - 3x = 0$ has at least two positive solutions.

why?

Example

$e^x - 3x = 0$ has at least two positive solutions.

Why?

Let $f(x) = e^x - 3x$, a continuous function on $\mathbb{R}$

$f(0) = 1$, $f(1) = e - 3 < 0$, $f(2) = e^2 - 6 > 0$

$\therefore$ Bolzano's thm $\Rightarrow$ $\exists$ solution in $[0,1]$

and $\exists$ solution in $[1,2]$

Example

For every $a \in [-1, 1]$, $\sin x = a$ has a solution between $-\pi/2$ and $\pi/2$

since $\sin(-\frac{\pi}{2}) = -1$ and $\sin(\pi/2) = 1$,

by IVT, for each $a \in [-1, 1]$ $\exists\, c \in [-\pi/2, \pi/2]$ s.t. $\sin c = a$.

Example

For any given $n \in \mathbb{N}$,

every positive real number has a positive n^{th} root.

That is, for every $a > 0$, $\exists\, b > 0$ s.t. $b^n = a$

n^{th} positive root of a.

Proof as HW

Apply IVT.