MATH 400

Real Analysis

Part #26

Lets recall a few ϵ-δ proofs of continuity of f at c

① $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$

In main step of the proof, we estimate

$$|f(x) - f(c)| = |x - c| < \epsilon \quad \text{for } |x-c| < \delta \text{ when } \boxed{\delta = \epsilon}$$

② $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 5x+2$

$$|f(x) - f(c)| = |5x+2-(5c+2)| = |5x-5c| = 5|x-c| < \epsilon$$

$$\text{for } |x-c| < \delta \text{ when } \boxed{\delta = \frac{\epsilon}{5}}$$

In both these examples, the choice of δ does <u>not</u> depend on c, the point at which continuity is desired.

③ $f: (0,\infty) \to \mathbb{R}$ with $f(x) = x^2$.

$|f(x) - f(c)| = |x^2 - c^2| = |x-c||x+c| < \epsilon$ for $|x-c| < \delta$

if $\delta < \dfrac{\epsilon}{|x+c|}$

cannot depend on the unknown variable x

When $\delta < 1$, then $|x-c| < 1 \Rightarrow |x+c| < 2c+1$

so we can choose $\delta < \dfrac{\epsilon}{2c+1}$ in that situation.

③ $f:(0,\infty)\to\mathbb{R}$ with $f(x)=x^2$.

$|f(x)-f(c)| = |x^2-c^2| = |x-c|\,|x+c| < \epsilon$ for $|x-c|<\delta$

if $\delta < \dfrac{\epsilon}{|x+c|}$

cannot depend on the unknown variable x

When $\delta<1$, then $|x-c|<1 \Rightarrow |x+c|<2c+1$

so we can choose $\delta<\dfrac{\epsilon}{2c+1}$ in that situation.

$\therefore$ the formal proof says.

Given $\epsilon>0$, choose $\boxed{\delta=\min\{1,\frac{\epsilon}{2c+1}\}}>0$

s.t. $|x-c|<\delta \Rightarrow |f(x)-f(c)| \le \dots\dots < \epsilon$

- choice of δ depends on c, the point at which continuity is desired.
- $\delta=\delta(\epsilon,c)$ because as c becomes larger we need smaller & smaller δ, its choice inversely proportional to c.

④ $\underline{f : [1\ 4] \to \mathbb{R}}$ with $f(x) = x^2$

$$|f(x) - f(c)| = |x^2 - c^2| = |x+c|\,|x-c| \leq 8\,|x-c| \quad \left(\begin{array}{l}\text{since } x, c \in [1,4] \\ \Rightarrow |x+c| \leq 8\end{array}\right)$$

$< \epsilon$

whenever $|x-c| < \delta$

for $\boxed{\delta = \dfrac{\epsilon}{8}}$

- Same function but on a different domain leads to $\delta = \delta(\epsilon)$, that does not depend on c.

We say a function is uniformly continuous if we can choose a $\delta = \delta(\epsilon)$ that does not depend on the point c.

Definition Let $f : A \to \mathbb{R}$. f is uniformly continuous on A if $\forall \epsilon > 0$ $\exists \delta > 0$ s.t. for all $x, y \in A$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Some examples of (uniformly?) continuous functions:

① $f(x) = 5x+2$ on $\mathbb{R}$ $\delta = \varepsilon/5$ works Uniformly Cont. on $\mathbb{R}$

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seems unavoidable

③ $f(x) = x^2$ on $[1,4]$ $\quad \delta = \frac{\epsilon}{8}$ works $\quad$ Unif. Cont. on $[1,4]$

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③ $f(x) = x^2$ on $[1,4]$ $\delta = \frac{\epsilon}{8}$ works Unif. Cont. on $[1,4]$

④ $f(x) = \sqrt{x}$ on $(0,\infty)$ $\delta = \epsilon^2$ works Unif. Cont. on $\mathbb{R}^+$

⑤ $f(x) = \sin x$ on $\mathbb{R}$ $\delta = \epsilon$ works Unif. Cont. on $\mathbb{R}$

⑥ $f(x) = \cos x$ on $\mathbb{R}$ $\delta = \epsilon$ works Unif. Cont. on $\mathbb{R}$

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⑥ $f(x) = \cos x$ on $\mathbb{R}$ $\quad \delta = \epsilon$ works $\quad$ Unif. Cont. on $\mathbb{R}$

⑦ $f(x) = 1/x$ on $(0,\infty)$ $\quad \delta = \delta(\epsilon, c)$ seems unavoidable $\quad$ Not Unif. Cont.?

⑧ $f(x) = 1/x$ on $(3,\infty)$ $\quad \delta = \delta(\epsilon)$ can be found $\quad$ Unif. Cont.

How to show f is not Unif. Cont. on A?

Prove the negation of the definition of Unif. Cont.

<u>Theorem [Sequential Criterion for Non-Unif. Cont.]</u>

$f: A \to \mathbb{R}$ fails to be Unif. Cont. on A if and only if $\exists\ \epsilon_0 > 0$ and $(x_n), (y_n) \subseteq A$ such that

$$|x_n - y_n| \to 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0$$

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Proof. f not unif cont. on $A \iff \exists\ \epsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists\ x, y \in A$ s.t. $|x-y| < \delta$ but $|f(x) - f(y)| \geq \epsilon_0$.

Set $\delta_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$, so $\exists\ x_n, y_n \in A$ s.t. $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$.

example $h(x) = \sin\left(\frac{1}{x}\right)$ is continuous on $(0,1)$
But not uniformly cont. on $(0,1)$

Take $\epsilon_0 = 2$ & set $x_n = \dfrac{1}{\pi/2 + 2n\pi}$, $y_n = \dfrac{1}{3\pi/2 + 2n\pi}$

Both x_n & $y_n \to 0$, so $|x_n - y_n| \to 0$

and $|h(x_n) - h(y_n)| = \left|\sin\left(\frac{\pi}{2} + 2n\pi\right) - \sin\left(\frac{3\pi}{2} + 2n\pi\right)\right| = 2$

$\forall n \in \mathbb{N}$.

Theorem [Uniform Continuity on Compact Sets]

If $f : A \to \mathbb{R}$ is continuous on A and A is compact then f is uniformly continuous on A.

Proof

See the textbook proof that uses "Sequential Compactness" of A.

Assuming f is not unif. cont., using the previous theorem we get (x_n) & (y_n) s.t. $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$

But sequential compactness of A applied to (x_n) & the continuity of f will lead to $|f(x_n) - f(y_n)| \to 0$.

Proof (using open cover defn. of compactness)

Let f be continous on compact set A.

Given $\epsilon > 0$, for each $c \in A$, since f is cont. on c,

$\exists\, \delta = \delta(c) > 0$ s.t. $x \in A$ and $|x - c| < \delta(c)$

to indicate δ might depend on c

$\Rightarrow |f(x) - f(c)| < \frac{\epsilon}{2}$

We want to show

Given $\epsilon > 0$, $\exists\, \delta > 0$ s.t. $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

for all $x, y \in A$

should only depend on ϵ

Proof (using open cover defn. of compactness)

Let f be continous on compact set A.

Given $\epsilon > 0$, for each $c \in A$, since f is cont. on c, $\exists\, \delta = \delta(c) > 0$ s.t. $x \in A$ and $|x - c| < \delta(c)$

to indicate δ might depend on c

$\Rightarrow |f(x) - f(c)| < \frac{\epsilon}{2}$

Note $\left\{ \left(c - \frac{\delta(c)}{2}, c + \frac{\delta(c)}{2} \right) : c \in A \right\}$ forms an open cover of A

By compactness of A, there is a finite subcover,

$A \subseteq \left\{ \left(c_1 - \frac{\delta(c_1)}{2}, c_1 + \frac{\delta(c_1)}{2} \right), \left(c_2 - \frac{\delta(c_2)}{2}, c_2 + \frac{\delta(c_2)}{2} \right), \ldots, \left(c_k - \frac{\delta(c_k)}{2}, c_k + \frac{\delta(c_k)}{2} \right) \right\}$

Set $\delta = \min \left\{ \frac{\delta(c_1)}{2}, \frac{\delta(c_2)}{2}, \ldots, \frac{\delta(c_k)}{2} \right\}$

note δ is not dependent on arbitrary c, it just a fn. of ϵ.

$$\delta = \min\left\{ \frac{\delta(c_1)}{2}, \frac{\delta(c_2)}{2}, \ldots, \frac{\delta(c_k)}{2} \right\}$$

For any $x, y \in A$, since we have a finite subcover of A

$x \in \left(c_i - \frac{\delta(c_i)}{2}, c_i + \frac{\delta(c_i)}{2}\right)$ for some $i \in \{1, \ldots, k\}$

★<u>Reminder</u>: we are assuming $|x-y| < \delta$

& trying to show $|f(x) - f(y)| < \epsilon$

<u>Idea</u>: $|f(x) - f(y)| = |f(x) - f(c_i) + f(c_i) - f(y)|$

$\leq |f(x) - f(c_i)| + |f(y) - f(c_i)|$

But $x \in (c_i - \delta(c_i)/2, c_i + \delta(c_i)/2) \not\Rightarrow y \in \left(c_i - \frac{\delta(c_i)}{2}, c_i + \frac{\delta(c_i)}{2}\right)$

$$\delta = \min\left\{ \frac{\delta(c_1)}{2}, \frac{\delta(c_2)}{2}, \ldots, \frac{\delta(c_k)}{2} \right\}$$

For any $x, y \in A$, since we have a finite subcover of A

$$x \in \left(c_i - \frac{\delta(c_i)}{2}, c_i + \frac{\delta(c_i)}{2} \right) \text{ for some } i \in \{1, \ldots, k\}$$

i.e., $|c_i - x| < \frac{\delta(c_i)}{2}$

Also, $|x - y| < \delta \leq \frac{\delta(c_i)}{2}$.

$$\Rightarrow \left[\begin{aligned} |c_i - y| &= |c_i - x + x - y| \\ &\leq |c_i - x| + |x - y| \leq \frac{\delta(c_i)}{2} + \frac{\delta(c_i)}{2} \\ &= \delta(c_i) \end{aligned} \right.$$

So, $|c_i - y| \leq \delta(c_i)$.

$\star$ <u>Reminder</u>: we are assuming $|x - y| < \delta$ & trying to show $|f(x) - f(y)| < \epsilon$

<u>Idea</u>: $|f(x) - f(y)| = |f(x) - f(c_i) + f(c_i) - f(y)|$

$$\leq |f(x) - f(c_i)| + |f(y) - f(c_i)|$$

But $x \in (c_i - \delta(c_i)/2, c_i + \delta(c_i)/2) \not\Rightarrow y \in \left(c_i - \frac{\delta(c_i)}{2}, c_i + \frac{\delta(c_i)}{2}\right)$

so

$$\delta = \min\left\{ \frac{\delta(c_1)}{2}, \frac{\delta(c_2)}{2}, \ldots, \frac{\delta(c_k)}{2} \right\}$$

For any $x, y \in A$, since we have a finite subcover of A

$$x \in \left(c_i - \frac{\delta(c_i)}{2}, c_i + \frac{\delta(c_i)}{2} \right) \text{ for some } i \in \{1, \ldots, k\}$$

i.e., $|c_i - x| < \dfrac{\delta(c_i)}{2}$

Also, $|x - y| < \delta \leq \dfrac{\delta(c_i)}{2}$.

$$\Rightarrow |c_i - y| = |c_i - x + x - y| \leq |c_i - x| + |x - y| \leq \frac{\delta(c_i)}{2} + \frac{\delta(c_i)}{2} = \delta(c_i)$$

So,

$|c_i - y| \leq \delta(c_i)$.

Since both $|x - c_i| \leq \delta(c_i)$ & $|y - c_i| \leq \delta(c_i)$,

by definition of continuity of f at c_i using $\delta(c_i)$, we have

$$|f(x) - f(c_i)| < \epsilon/2 \quad \text{and} \quad |f(y) - f(c_i)| < \epsilon/2$$

$$\therefore |f(x) - f(y)| = |f(x) - f(c_i) + f(c_i) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

A Mathematical Theme $\rightarrow$

Local Property $\Rightarrow$ Global Property

If f is a continuous function on a compact interval

then
- f is bounded
- f has a maximum & a minimum
- f is uniformly continuous

"local property of f on $[a,b]$" $\Rightarrow$ "Global property of f on $[a,b]$"

MATH 400

Real Analysis

Part # 27

Definition [Differentiability]

Let $f : I \to \mathbb{R}$ be a function on an interval I.

Given $c \in I$, the derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$ provided the limit exists.

We say f is differentiable at c.

If f is differentiable at every $c \in I$, then f is differentiable on I.

Examples

① $f(x) = x^n$

Let $c \in \mathbb{R}$

$$f'(c) = \lim_{x \to c} \frac{x^n - c^n}{x - c} = \lim_{x \to c} \left(x^{n-1} + c x^{n-2} + c^2 x^{n-3} + \cdots + c^{n-1}\right)$$

$$= c^{n-1} + c^{n-1} + \cdots + c^{n-1} = n c^{n-1}$$

Examples

① $f(x) = x^n$

Let $c \in \mathbb{R}$

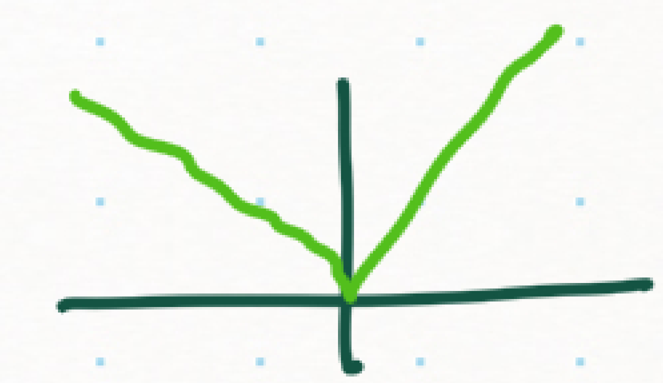
$x^n - c^n = (x-c)(x^{n-1} + cx^{n-2} + \cdots + c^{n-1})$

$$f'(c) = \lim_{x\to c} \frac{x^n - c^n}{x-c} = \lim_{x\to c}\left(x^{n-1}+cx^{n-2}+c^2x^{n-3}+\cdots+c^{n-1}\right)$$

$$= c^{n-1}+c^{n-1}+\cdots+c^{n-1} = n\,c^{n-1}$$

② $f(x)=|x|$, let $c=0$

Note f is continuous at 0 since $\lim_{x\to 0} f(x) = f(0)$



$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0} = \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{|x|}{x}$$

This limit does not exist

Examples

① $f(x) = x^n$

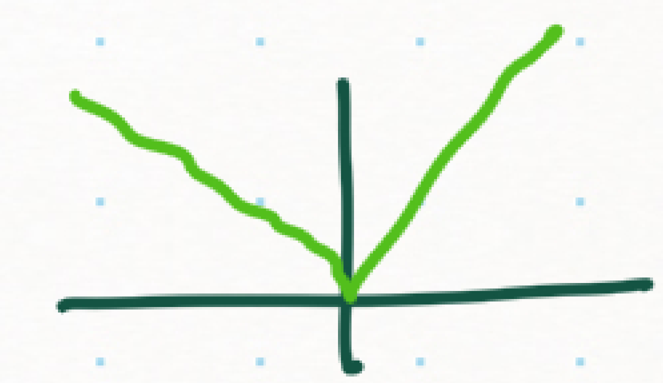
Let $c \in \mathbb{R}$

$x^n - c^n = (x-c)(x^{n-1} + cx^{n-2} + \cdots + c^{n-1})$

$$f'(c) = \lim_{x\to c} \frac{x^n - c^n}{x - c} = \lim_{x\to c} (x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1})$$

$$= c^{n-1} + c^{n-1} + \cdots + c^{n-1} = n c^{n-1}$$

② $f(x) = |x|$, let $c = 0$



Note f is continuous at 0 since $\lim_{x\to 0} f(x) = f(0)$

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{|x|}{x}$$

$(a_n) = \left(\frac{1}{n}\right) \to 0$

$(b_n) = \left(-\frac{1}{n}\right) \to 0$

But $\frac{|a_n|}{a_n} = 1 \to 1$

$\neq$

$\frac{|b_n|}{b_n} = -1 \to -1$

$\Longrightarrow$ This limit does not exist

We just saw continuity need not imply differentiability.

But

Theorem If $f: I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.

We just saw Continuity need not imply differentiability.

But

<u>Theorem</u> If $f: I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.

<u>Proof</u> We assume $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists

We want to show $\lim_{x \to c} f(x) = f(c)$.

By Alegra of functional limits,

$$\lim_{x \to c} (f(x) - f(c)) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \right) (x - c) = \lim \left(\frac{f(x) - f(c)}{x - c} \right) \lim (x - c)$$

$$= f'(c) \, (0) = 0$$

$\therefore \lim_{x \to c} f(x) = f(c)$

Algebra of Differentiable Functions

Let f, g be functions on an interval I. and assume f and g are differentiable at $c \in I$. Then,

① [Linearity-1] $(f+g)'(c) = f'(c) + g'(c)$

② [Linearity-2] $(kf)'(c) = kf'(c)$ for all $k \in \mathbb{R}$.

③ [Product Rule] $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

④ [Quotient Rule] $\left(f/g\right)'(c) = \dfrac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}$, $\forall\, g(c) \neq 0$

Proof of $(f+g)'(c) = f'(c) + g'(c)$

$$\lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x-c} = \lim_{x \to c} \frac{[f(x)+g(x)] - [f(c)+g(c)]}{x-c}$$

$$= \lim_{x \to c} \frac{[f(x)-f(c)] + [g(x)-g(c)]}{x-c}$$

allowed since each limit exists

$$= \lim_{x \to c} \frac{f(x)-f(c)}{x-c} + \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$$

$$= f'(c) + g'(c)$$

Proof $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

$$\frac{(fg)(x) - (fg)(c)}{x-c} = \frac{f(x)g(x) - f(c)g(c)}{x-c}$$

$$= \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x-c}$$

$$= f(x)\left[\frac{g(x)-g(c)}{x-c}\right] + g(c)\left[\frac{f(x)-f(c)}{x-c}\right]$$

When we take the limit $x \to c$

$f(x) \to f(c)$ since f is continuous at c

f is differentiable at c $\Rightarrow$ f is continuous at c

$\frac{g(x)-g(c)}{x-c} \to g'(c)$ assumed

$g(c) = g(c)$ fixed

$\frac{f(x)-f(c)}{x-c} \to f'(c)$ assumed

Theorem [Chain Rule] Let $f: I_1 \to \mathbb{R}$ and $g: I_2 \to \mathbb{R}$ such that $f(I_1) \subseteq I_2$ so that $g \circ f$ is defined. If f is differentiable at $c \in I_1$ and g is differentiable at $f(c) \in I_2$ then $(g \circ f)'(c) = g'(f(c))\, f'(c)$.

Proof

__Theorem [Chain Rule]__ Let $f: I_1 \to \mathbb{R}$ and $g: I_2 \to \mathbb{R}$ such that $f(I_1) \subseteq I_2$ so that $g \circ f$ is defined. If f is differentiable at $c \in I_1$ and g is differentiable at $f(c) \in I_2$ then $(g \circ f)'(c) = g'(f(c))\, f'(c)$.

__Proof__ Consider the function
$$h(y) = \begin{cases} \left(\dfrac{g(y) - g(f(c))}{y - f(c)}\right) & \text{if } y \neq f(c) \\ g'(f(c)) & \text{if } y = f(c) \end{cases}$$

take lim $y \to f(c)$

Note h is continuous at $f(c)$, i.e., $\lim_{y \to f(c)} h(y) = g'(f(c))$

since g is differentiable at $f(c)$.

Claim $$\frac{g(f(x))-g(f(c))}{x-c} = h(f(x))\,\frac{f(x)-f(c)}{x-c}$$

If $f(x) \neq f(c)$ then

$$\frac{g(f(x))-g(f(c))}{x-c} = \overbrace{\frac{g(f(x))-g(f(c))}{f(x)-f(c)}}^{h(f(x))}\,\frac{f(x)-f(c)}{x-c}$$

If $f(x) = f(c)$ then

$$LHS = \frac{g(f(x))-g(f(c))}{x-c} = \frac{0}{x-c} = 0$$

$$RHS = g'(f(c))\,\frac{f(x)-f(c)}{x-c} = g'(f(c))\,\frac{0}{x-c} = 0$$

Claim $\dfrac{g(f(x)) - g(f(c))}{x-c} = h(f(x)) \dfrac{f(x) - f(c)}{x-c}$

To finish the proof.

$$(g \circ f)'(c) = \lim_{x \to c} \frac{g(f(x)) - g(f(c))}{x-c}$$

$$= \lim_{x \to c} h(f(x)) \frac{f(x) - f(c)}{x-c}$$

$$= \left(\lim_{x \to c} h(f(x))\right)\left(\lim_{x \to c} \frac{f(x) - f(c)}{x-c}\right)$$

$$= g'(f(c))\, f'(c).$$

Topologist's sine curve

$$g_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

① $g_0(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

we have discussed $g_0(x)$ is not continuous at 0.

& hence not differentiable at 0

② $g_1(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Since $\sin(1/x) \in [-1, 1]$, we have $-|x| \leq x\sin(1/x) \leq |x|$

$$\underbrace{-|x|}_{\downarrow\, 0} \leq x\sin(1/x) \leq \underbrace{|x|}_{\downarrow\, 0} \quad \text{as } x \to 0$$

By Squeeze Thm.,

$$\lim_{x\to 0} x\sin(1/x) = 0$$

$$\therefore \lim_{x\to 0} g_1(x) = g_1(0)$$

& g_1 is continuous at 0

But g_1 is not differentiable at 0

$$\lim_{x\to 0} \frac{g_1(x) - g_1(0)}{x - 0} = \lim_{x\to 0} \frac{x\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} \sin(1/x)$$ which does not exist

③ $g_2(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Same argument as $g_1(x)$ shows $g_2(x)$ is continuous at 0

g_2 is also differentiable at 0:

$$g_2'(0) = \lim_{x \to 0} \frac{g_2(x) - g_2(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x \to 0} x \sin(1/x) = 0$$

What is $g_2'(x)$?

By rules of Differentiation we get

$$g_2'(x) = \begin{cases} -\cos(1/x) + 2x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

g_2' exists everywhere on $\mathbb{R}$ but g_2' is not continuous

($\lim_{x \to 0} g_2'(x)$ does not exist)

MATH 400

Real Analysis

Part #28

Applications of Differential Calculus are based on finding max/min values of $f(x)$ by ~~solving~~ $f'(x)=0$.

Theorem [Interior Extremum Theorem / Fermat's Thm]

Let f be differentiable on (a,b).

If f attains a max value at $c \in (a,b)$

(i.e., $f(c) \geq f(x)\ \forall x \in (a,b)$)

then $f'(c)=0$

Same is true for $f(c)$ is min value.

Proof $[f(c) \geq f(x)\ \forall x \in (a,b) \Rightarrow f'(c) = 0]$

Since $c \in (a,b)$, an open interval, we can find $(x_n), (y_n) \subseteq (a,b)$ s.t. $c - \frac{1}{n} < x_n < c$ and $c < y_n < c + \frac{1}{n}$ $\forall n$.

Since $f(c)$ is a maximum, $f(y_n) - f(c) \leq 0$ $\forall n$

and $y_n - c \geq 0$ $\forall n$

which implies $\boxed{\dfrac{f(y_n) - f(c)}{y_n - c} \leq 0}$ $\forall n$

By Order Limit Thm, $f'(c) = \lim_{n \to \infty} \dfrac{f(y_n) - f(c)}{y_n - c} \leq 0$

since $y_n \to c$

Proof $[f(c) \geq f(x)\ \forall x \in (a,b) \Rightarrow f'(c) = 0]$

Since $c \in (a,b)$, an open interval, we can find $(x_n), (y_n) \subseteq (a,b)$ s.t. $c - \frac{1}{n} < x_n < c$ and $c < y_n < c + \frac{1}{n}$ $\forall n$.

Since $f(c)$ is a maximum, $f(y_n) - f(c) \leq 0$ $\forall n$
and $y_n - c \geq 0$ $\forall n$

which implies $\dfrac{f(y_n) - f(c)}{y_n - c} \leq 0$ $\forall n$

By Order Limit Thm, $f'(c) = \lim\limits_{n \to \infty} \dfrac{f(y_n) - f(c)}{y_n - c} \leq 0$ — ①

Also $f(x_n) - f(c) \leq 0$ $\forall n$ and $x_n - c \leq 0$ $\forall n$

which implies $\dfrac{f(x_n) - f(c)}{x_n - c} \geq 0$ $\forall n$

Again by OL Thm, $f'(c) = \lim \dfrac{f(x_n) - f(c)}{x_n - c} \geq 0$ — ②

① & ② $\Rightarrow f'(c) = 0$

We know: f differentiable $\Rightarrow f$ continuous $\Rightarrow f$ has IVP

Intermediate Value Property

What about the function f'?

we know f' exists $\not\Rightarrow (f')'$ exists

we know f' exists $\not\Rightarrow f'$ continuous

Does f' have the IVP??

We know: f differentiable $\Rightarrow$ f continuous $\Rightarrow$ f has IVP

Intermediate Value Property

What about the function f'?

we know f' exists $\not\Rightarrow$ $(f')'$ exists

we know f' exists $\not\Rightarrow$ f' continuous

Theorem [Darboux' Theorem]

Let f be differentiable on $[a,b]$.

If α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(b) < \alpha < f'(a)$)

then $\exists\, c \in (a,b)$ s.t. $f'(c) = \alpha$.

We know: f differentiable $\Rightarrow$ f continuous $\Rightarrow$ f has IVP

Intermediate Value Property

What about the function f'?

We know f' exists $\not\Rightarrow$ $(f')'$ exists

We know f' exists $\not\Rightarrow$ f' continuous

Theorem [Darboux' Theorem]

Let f be differentiable on $[a,b]$.

If α satisfies $f'(a)<\alpha<f'(b)$ (or $f'(b)<\alpha<f'(a)$)

then $\exists\, c\in(a,b)$ s.t. $f'(c)=\alpha$.

Proof WLOG assume $f'(a)<\alpha<f'(b)$

Define $\boxed{g(x)=f(x)-\alpha x}$

Then g is differentiable on $[a,b]$ & $g'(x)=f'(x)-\alpha$

Prog $\left[f'(a) < \alpha < f'(b) \Rightarrow \exists\, c \in (a,b) \text{ s.t. } f'(c) = \alpha \right]$

$g(x) = f(x) - \alpha x$ & $g'(x) = f'(x) - \alpha$ (why?)

$g'(a) = f'(a) - \alpha < 0$ $\qquad$ $g'(b) = f'(b) - \alpha > 0$

We want $c \in (a,b)$ s.t. $g'(c) = 0$ i.e., $f'(c) = \alpha$

Step 1a [HW] $\exists\, c_1 \in (a,b)$ s.t. $g(c_1) < g(a)$ (i.e. $g(a)$ is not a min)

Step 1b [HW] $\exists\, c_2 \in (a,b)$ s.t. $g(c_2) < g(b)$ (i.e. $g(b)$ is not a min)

Step 2 [HW] $\exists\, c \in (a,b)$ s.t. $g'(c) = 0$

Prog $\left[\, f'(a) < \alpha < f'(b) \Rightarrow \exists\, c \in (a,b) \text{ s.T. } f'(c) = \alpha \,\right]$

$g(x) = f(x) - \alpha x$ & $g'(x) = f'(x) - \alpha$ (why?)

$g'(a) = f'(a) - \alpha < 0$ $\qquad$ $g'(b) = f'(b) - \alpha > 0$

We want $c \in (a,b)$ s.T. $g'(c) = 0$ i.e., $f'(c) = \alpha$

Step 1a [HW] $\exists\, c_1 \in (a,b)$ s.T. $g(c_1) < g(a)$

$g'(a) = \lim_{n\to\infty} \dfrac{g(x_n) - g(a)}{x_n - a} < 0$ choosing $x_n \geq a$ means numerator must become negative

Step 1b [HW] $\exists\, c_2 \in (a,b)$ s.T. $g(c_2) < g(b)$

— same idea a step 1a. —

Step 2 [HW] $\exists\, c \in (a,b)$ s.T. $g'(c) = 0$

what does Extreme Value Thm tell us about g in $[a,b]$?

what does Interior Extremum Thm tell us about g' in (a,b)?