Math 554

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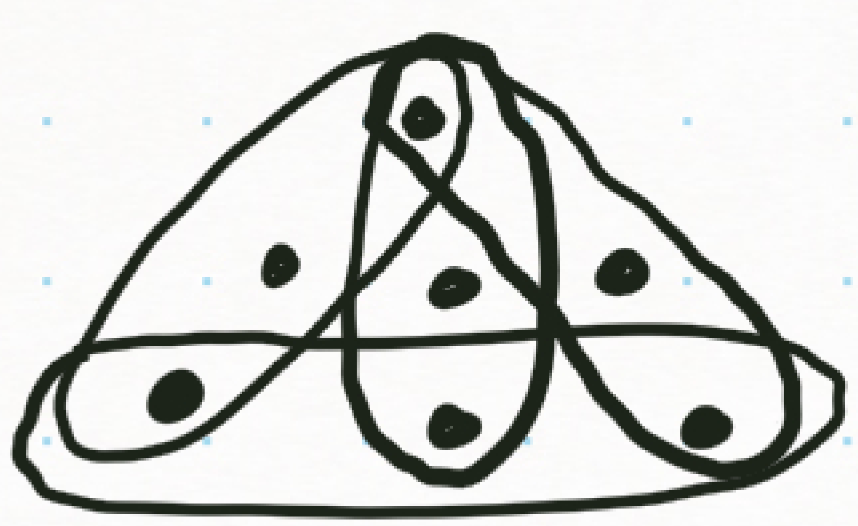
Counting the number of "copies" of hypergraph H in G

To understand this problem, we need to specify what "copy" means:

	Graphs	Hypergraphs
• Allow removal of both vertices & edges	Subgraph	Partial hypergraph
• Allow removal of only vertices	Induced subgraph	Two possibilities What to do with edges whose at least one vertex has been removed?



"Shrink" the edge but keep it.

Remove any such edge

Confusingly, both versions are called subhypergraph in the literature.

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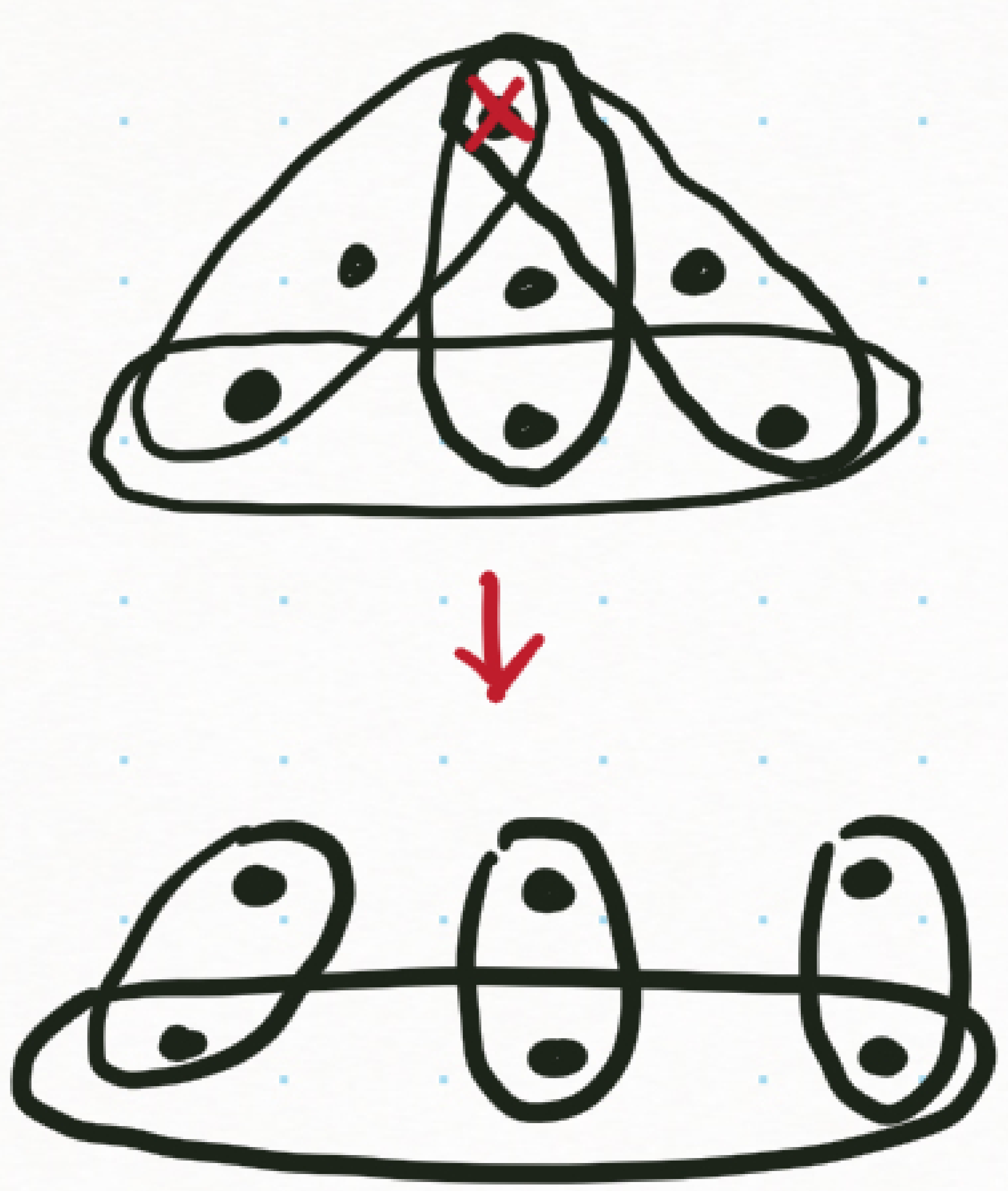

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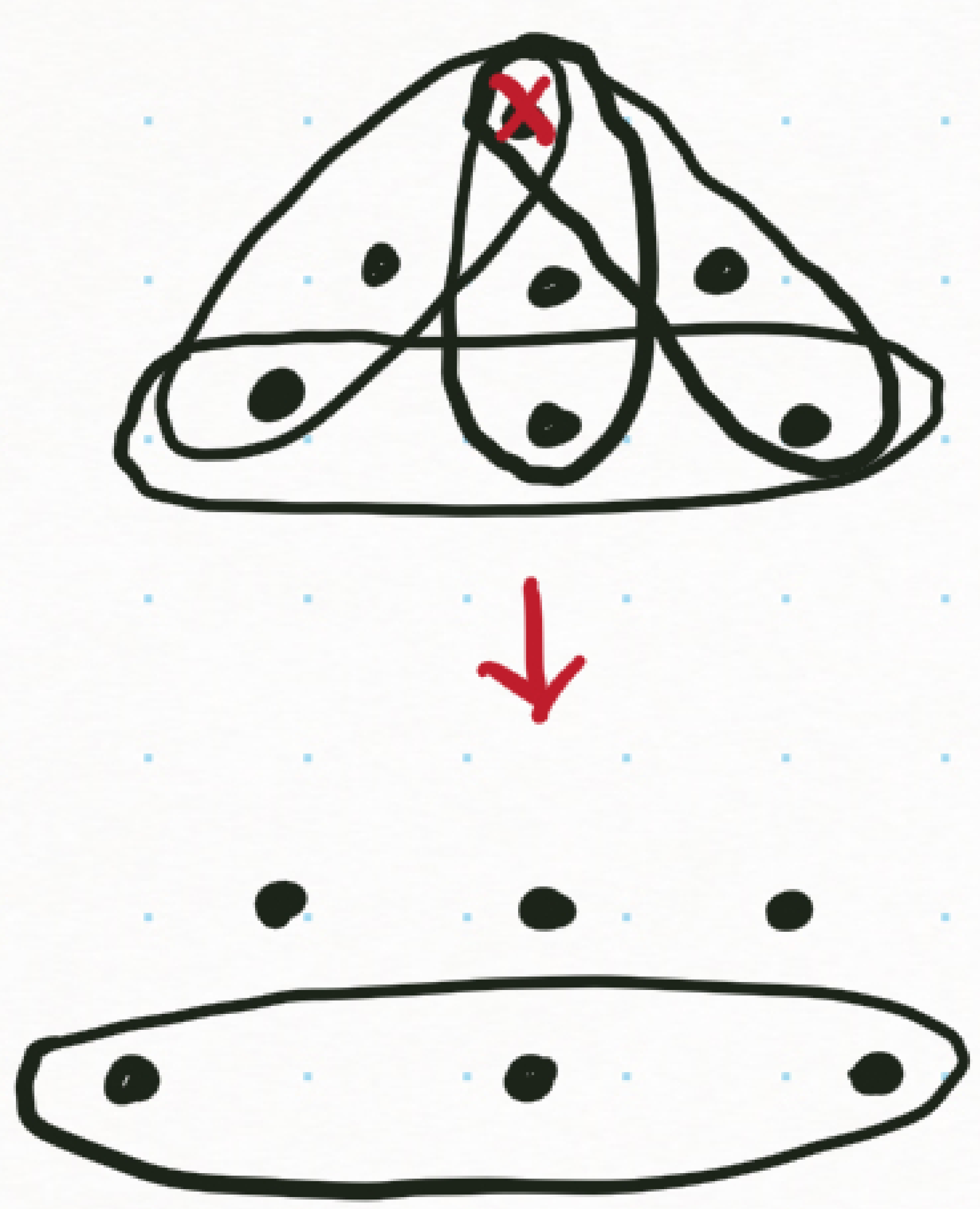

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★ $\boxed{\text{Remove any such edge}}$

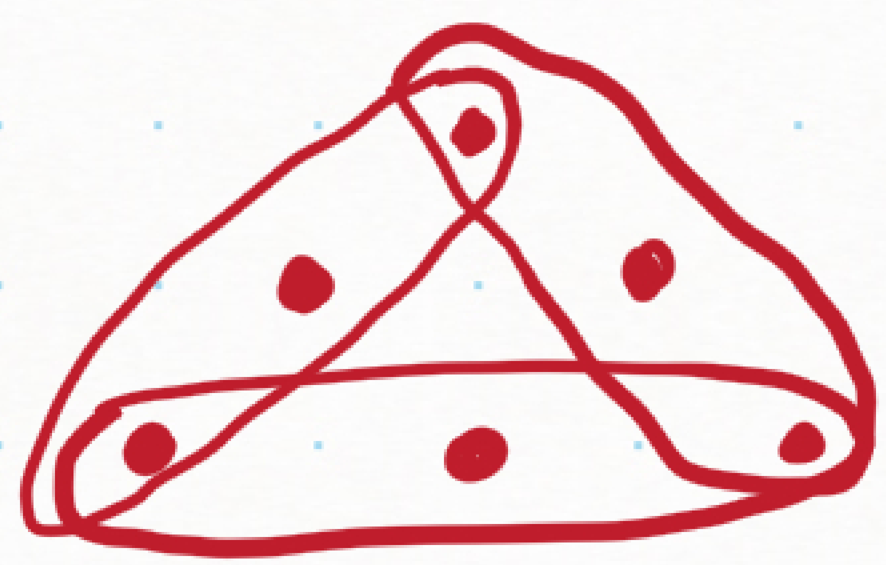
Defn $H=(V',E')$ is a subhypergraph of $G=(V,E)$ induced by $V'\subseteq V$ if $E'=\{e\in E : e\subseteq V'\}$

Confusingly, both versions are called subhypergraph in the literature.

is the definition we will use.

Defn A <u>hypergraph homomorphism</u> $\varphi: V(H) \to V(G)$ is a map which preserves edges, i.e. each edge of H maps to an edge of G.

A <u>copy of H in G</u> corresponds to an injective homomorphism from H to G, i.e., a 1-to-1 map $\sigma: V(H) \to V(G)$ s.t. $\sigma(e) \in E(G)$ $\forall e \in E(H)$.



H

G

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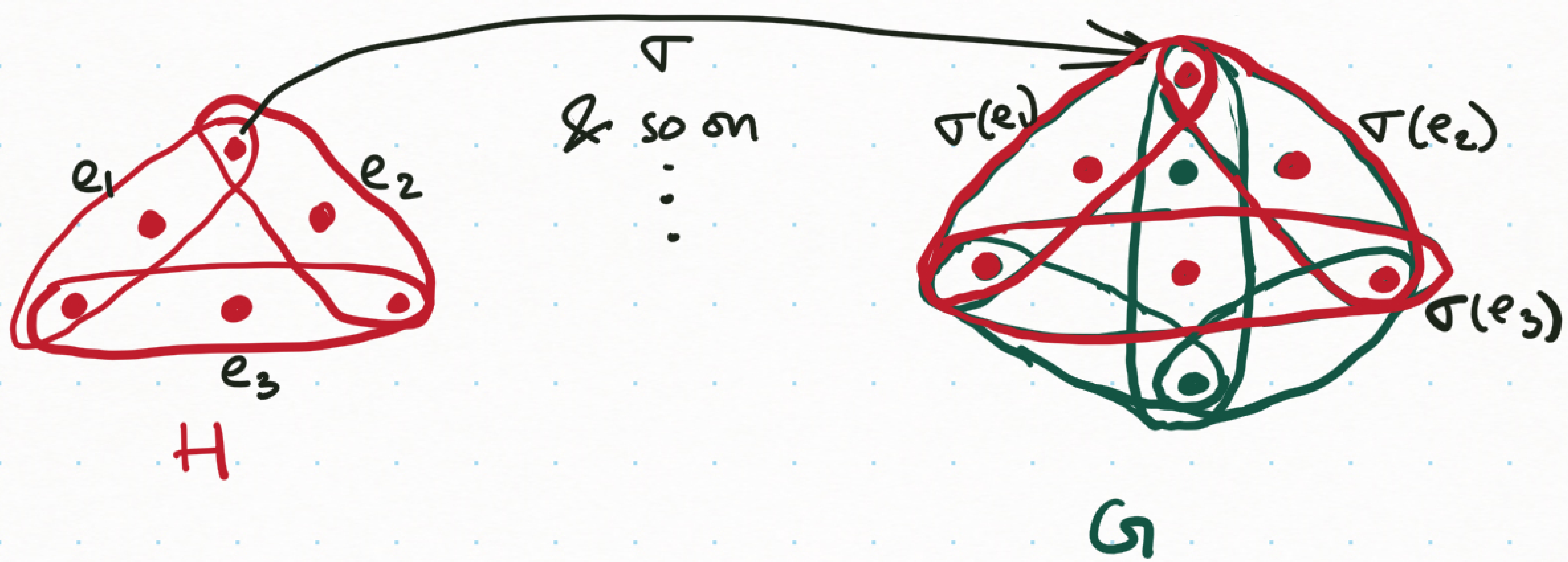

There is a copy of H in G.

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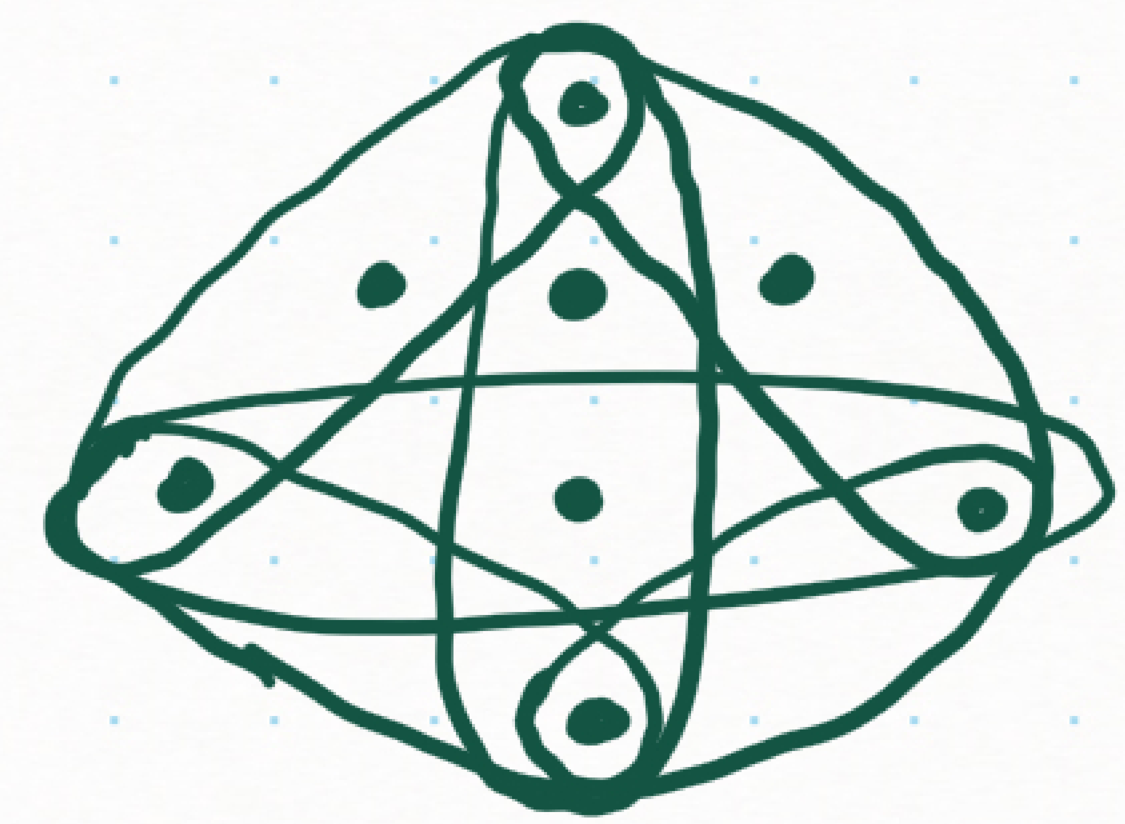

G

Is there a copy of this H in G?

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Defn $N(H, \ell)$ = max # copies of a fixed hypergraph H that can appear in a hypergraph G with ℓ edges.

H fixed, small
ℓ large

Theorem [Friedgut & Kahn 1998]

For a hypergraph H with fractional cover number $\rho^*(H)$, $\exists\, c_1, c_2$ s.t. $c_1 \ell^{\rho^*(H)} \le N(H, \ell) \le c_2 \ell^{\rho^*(H)}$ for all ℓ.

Linear Programming Duality

$$\left.\begin{aligned} \max\ & c^T x \\ \text{s.t. } & Ax \leq b \\ & x \geq 0 \end{aligned}\right\} 1 \quad \xleftrightarrow{\text{Dual}} \quad \left.\begin{aligned} \min\ & b^T y \\ \text{s.t. } & A^T y \geq c \\ & y \geq 0 \end{aligned}\right\} 2$$

where
$x \in \mathbb{R}^n$
$c \in \mathbb{R}^n$
$b \in \mathbb{R}^m$
$A_{m \times n}$

Weak Duality says $c^T x$ for any feasible solution x of 1
is always at most $b^T y$ for any feasible solution y of 2.

Strong Duality of Linear Programming says
If x^* is an optimal solution of 1 and y^* is an optimal soln. of 2
then $c^T x^* = b^T y^*$.

A Binary Linear Program is a linear program whose variables are restricted to be 0 or 1.

Replacing $x \in \{0, 1\}$ by $x \in [0, 1]$ gives the Linear Program relaxation.

Independence number and cover number of hypergraphs

Defn An independent set of a hypergraph H is a subset of $V(H)$ that meets each edge at most once.

$\alpha(H) =$ max size of an ind. set of H = independence number of H

$$= \max_{\varphi} \sum_{v \in V(H)} \varphi(v)$$ where the max is taken over all functions $\varphi : V(H) \to \{0,1\}$ s.t. $\sum_{v \in e} \varphi(v) \leq 1 \;\; \forall e \in E(H)$

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<u>Independence function of H.</u>

Defn A <u>fractional independent set</u>, or <u>fractional independence function of H</u> is $\varphi : V(H) \to [0,1]$ s.t. $\sum_{v \in e} \varphi(v) \le 1 \ \forall e \in E(H)$

$\alpha^*(H) =$ fractional independence number of H

$= \max_{\varphi} \sum_{v \in V(H)} \varphi(v)$ where the max is taken over all fractional independence functions of H.

Note: $\alpha(H) \le \alpha^*(H)$

Independence number and cover numbers of hypergraphs

Defn An (edge) cover of a hypergraph H is a set of edges whose union gives all of $V(H)$, that is each vertex is contained in at least one chosen edge.

$\rho(H)$ = min size of a cover of H = (edge) cover number of H

$= \min\limits_{\varphi} \sum\limits_{e \in E(H)} \varphi(e)$ where the min is taken over all functions $\boxed{\varphi : E(H) \to \{0,1\} \text{ s.t. } \sum\limits_{e : v \in e} \varphi(e) \geq 1 \ \forall v \in V(H)}$

cover function of H

Defn A fractional (edge) cover, or fractional cover function of H is $\varphi : E(H) \to [0,1]$ s.t. $\sum\limits_{e : v \in e} \varphi(e) \geq 1 \ \forall v \in V(H)$

$\rho^*(H)$ = fractional cover number of H

$= \min\limits_{\varphi} \sum\limits_{e \in E(H)} \varphi(e)$ where the min is taken over all fractional cover functions of H.

Note: $\rho^*(H) \leq \rho(H)$

Recall, $\alpha(H) \leq \rho(H)$ (an edge cover has to use a different edge for each vertex of an independent set)

By Strong Duality, $\boxed{\alpha^*(H) = \rho^*(H)}$

$$\left.\begin{aligned} \alpha^*(H) = \max_{\varphi} & \sum_{v \in V(H)} \varphi(v) \\ \text{s.t.} & \\ & \sum_{v \in e} \varphi(v) \leq 1 \quad \forall e \in E(H) \\ & 0 \leq \varphi(v) \leq 1 \quad \forall v \in V(H) \end{aligned}\right] \xleftrightarrow{\text{Dual}} \left[\begin{aligned} \rho^*(H) = \min_{\psi} & \sum_{e \in E(H)} \psi(e) \\ \text{s.t.} & \\ & \sum_{e:\, v \in e} \psi(e) \geq 1 \quad \forall v \in V(H) \\ & 0 \leq \psi(e) \leq 1 \quad \forall e \in E(H) \end{aligned}\right.$$

Proof of $N(H,\ell) \le C_2\, \ell^{\rho^*(H)}$

Let G be any hypergraph w. ℓ edges

Assume each edge of H has at most k vertices.

Let $N = N(H,G) = \#$ copies of H in G.

Let $\Sigma = \{\sigma : V(H) \to V(G)$, 1-1 homomorphisms$\}$, so $N = |\Sigma|$.

We think of $\sigma \in \Sigma$ as $(\sigma(v) : v \in H)$, i.e. $(\sigma(v_1), \sigma(v_2), \dots, \sigma(v_n))$.

Let $\psi : E(H) \to [0,1]$ be an optimal fractional cover of H,

that is $\sum_{e \in E(H)} \psi(e) = \rho^*(H)$.

We may assume $\rho^*(H) \in \mathbb{Q}$, so $\rho^*(H) = \frac{s}{t}$ s.t. $\psi(e) = \frac{w(e)}{t}$ $\forall e$

where $w(e) \in \mathbb{Z}$ and t as common denominator of all $\psi(e)$

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Let G be any hypergraph w. ℓ edges

Assume each edge of H has at most k vertices.

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Let $\Sigma_1 = \{\sigma : V(H) \to V(G)$, 1-1 homomorphisms$\}$, so $N = |\Sigma_1|$.

We think of $\sigma \in \Sigma_1$ as $(\sigma(v) : v \in H)$, i.e. $(\sigma(v_1), \sigma(v_2), \dots, \sigma(v_n))$.

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Let $\{e_1, e_2, \dots, e_m\}$ be a list (a multiset) of edges of H where each edge of H appears exactly $w(e)$ times.

Since $\sum_{e : v \in e} \psi(e) \ge 1 \;\; \forall v \Rightarrow \sum_{e : v \in e} w(e) \ge t \;\; \forall v$, each vertex appears in at least t edges in this list.

Let σ be chosen uniformly at random from Σ

so, $\log|\Sigma| = H(\sigma) = H(\sigma(v_1), \ldots, \sigma(v_n)) \leq \sum_{i=1}^{n} H(\sigma(v_i))$

$\hookrightarrow$ trivial upper bd.

<u>Apply Shearer's lemma</u>

What should we project σ onto ?

$$H(\sigma) \leq \frac{1}{k} \sum_{F \in \mathcal{F}} H(\sigma_F)$$

$\mathcal{F} = ?$ & $k = ?$ for the $\mathcal{F}$.

$\mathcal{F} = \{e_1, \ldots, e_m\}$ works with $k = t$.

Think! Go back 1 page.

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By Shearer $\leq \frac{1}{t} \sum_{i=1}^{m} H(\sigma_{e_i})$, where $\sigma_{e_i} = (\sigma(v) : v \in e_i)$

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$\leq \frac{1}{t} \sum_{i=1}^{m} (\log(k! * \ell))$ since $H(\sigma_{e_i}) = H(\sigma(v) : v \in e_i)$

$\leq \log(k! * \ell)$

\# choices of mapping vertices in each edge from H to G

\# choices for edges

$(-,-,-,-,-) \xrightarrow{\sigma} (-,-,-,-,-)$

$e \in H$ $\qquad$ $\sigma(e) \in G$

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\# choices of mapping vertices in each edge from H to G

\# choices for edges

$(, , , ,) \xrightarrow{\sigma} (, , , ,)$

$e \in H$ $\quad$ $\sigma(e) \in G$

$\leq \frac{1}{t} \log(k! * l) \sum_{i=1}^{m} 1$

$= \frac{1}{t} \log(k! * l) \sum_{e \in E(H)} \omega(e)$ since $\{e_1, \dots, e_m\}$ contains each edge $\omega(e)$ times

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$\leq \log(\underset{\substack{\uparrow \\ \text{\# choices of mapping vertices in each edge from H to G}}}{k!} * \underset{\substack{\uparrow \\ \text{\# choices for edges}}}{\ell})$

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$= \log(k! * \ell) \sum_{e \in E(H)} \psi(e)$ since $\psi(e) = \frac{w(e)}{t}\ \forall e$

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$= \log(k! * l) \sum_{e \in E(H)} \psi(e)$ since $\psi(e) = \frac{\omega(e)}{t} \; \forall e$

$= \log(k! \, l) \, \rho^*(H)$

$\therefore |\Sigma| \leq (k! \, l)^{\rho^*(H)} = \underbrace{(k!)^{\rho^*(H)}}_{} \, l^{\rho^*(H)} = c_2 \, l^{\rho^*(H)}$

where c_2 depends only on H.

Proof of $N(H,\ell) \geq c_1 \ell^{\rho^*(H)}$

We have to, for each ℓ, find/construct G with at most ℓ edges that contains many $(c_1 \ell^{\rho^*(H)})$ copies of H.

Proof of $N(H,\ell) \geq c_1 \ell^{\varphi^*(H)}$

We have to, for each ℓ, find/construct G with at most ℓ edges that contains many ($c_1 \ell^{\varphi^*(H)}$) copies of H.

Let φ^* be an optimal fractional independent with size $\alpha^*(H)$.

Let $\ell_1 = \ell / |E(H)|$, and $V(H) = \{v_1, \ldots, v_n\}$.

For each $i \in [n]$, let X_i be a set of $\ell_1^{\varphi^*(v_i)}$ vertices s.t. X_is are pairwise disjoint.

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(Note $|X_i|$ as defined may not be integer, ~~to~~ to be more precise we have to be formal with $\ell_1 \in \mathbb{Q}$ and $\ell_1^{\varphi^*(v_i)} \in \mathbb{Q}$, and taking common denominators, & so on. But this just obfuscates the very simple underlying idea — Blow up H appropriately)

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Set $V(G) = \bigcup_{i=1}^{n} X_i$.

G will be an n-partite hypergraph as follows:

$\{x_{i_1}, x_{i_2}, \dots, x_{i_s}\}$ is an edge in G with $x_{i_j} \in X_{i_j}$ iff $\{v_{i_1}, v_{i_2}, \dots, v_{i_s}\}$ is an edge in H.

each edge in H gives us multiple edges in G with each vertex of X_i acting like $v_i \in V(H)$

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G contains at most $\prod_{v \in e} \ell_1^{\varphi^*(v)}$ copies of any edge e in H

$\therefore$ G has at most $|E(H)| \times \prod_{v \in e} \ell_1^{\varphi^*(v)} = |E(H)|\, \ell_1^{\sum_{v \in e} \varphi^*(v)} \leq |E(H)|\, \ell_1^{1} \leq \ell$ edges

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Also, G contains at least $\prod_{i=1}^{n} |X_i| = \prod \ell_1^{\varphi^*(v)} = \ell_1^{\sum_{v \in V(H)} \varphi^*(v)} = \ell_1^{\alpha^*(H)} = \ell_1^{\rho^*(H)} =$
$= \left(\frac{1}{|E(H)|}\right)^{\rho^*(H)} \ell^{\rho^*(H)} = c_1 \ell_1^{\rho^*(H)}$, where $c_1 = \left(\frac{1}{|E(H)|}\right)^{\rho^*(H)}$, copies of H ■

A Fundamental Enumerative Question

Fix H.

For a family $\mathcal{G}$ of graphs, which $G \in \mathcal{G}$ maximizes $|\mathrm{Hom}(G,H)|$ where $\mathrm{Hom}(G,H) = \{\text{homomorphisms from } G \text{ to } H\}$.

e.g. If $H = K_q$, then $\mathrm{Hom}(G,H) =$ all possible proper q-colorings of G.

If $H = H_{ind} =$  then $\mathrm{Hom}(G,H) =$ all possible independent sets in G

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If $H = H_{ind} =$  then $\mathrm{Hom}(G,H) =$ all possible independent sets in G $= \mathcal{I}(G)$

Theorem [Kahn 2001; partially solving a conjecture of Alon 1991]

Let $G = (A,B;E)$ be a d-regular bipartite graph with bipartition A, B where $|A| = |B| = N$, then $|\mathcal{I}(G)| \leq |\mathcal{I}(K_{d,d})|^{N/d} = (2^{d+1}-1)^{N/d}$

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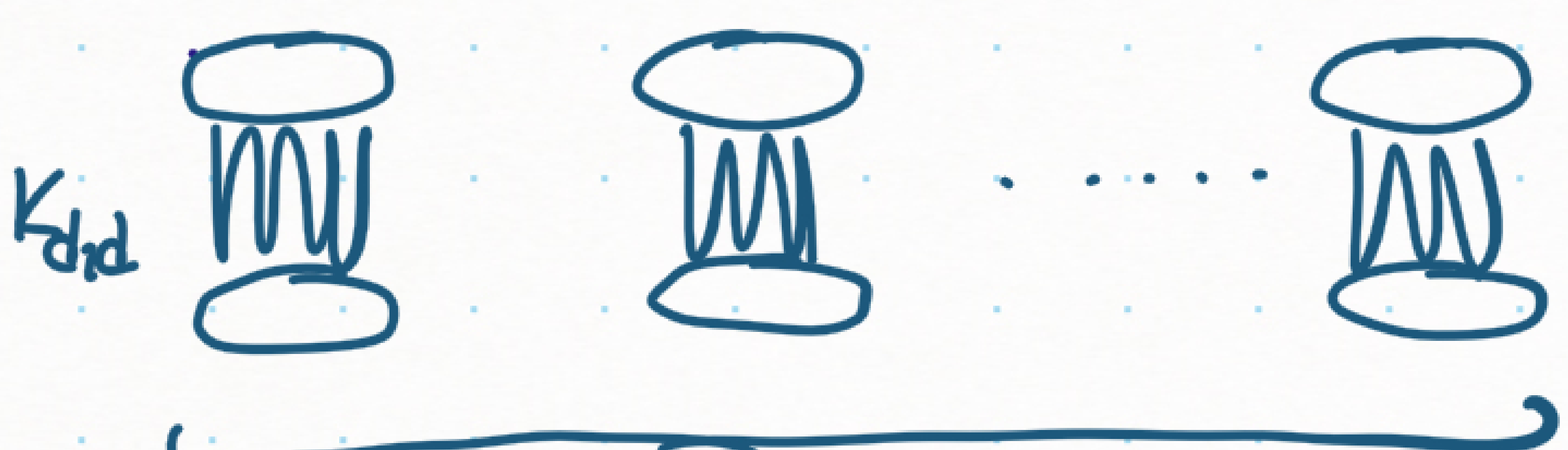

N/d copies of $K_{d,d}$ has $(2^{d+1}-1)^{N/d}$ independent sets.

(2^d choices in each partite set -1 for empty set counted twice)

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If $H = H_{ind} =$  then $\mathrm{Hom}(G,H) =$ all possible independent sets in G $= \mathcal{I}(G)$

Theorem [Kahn 2001; partially solving a conjecture of Alon 1991]
Let $G = (A, B; E)$ be a d-regular bipartite graph with bipartition A, B where $|A| = |B| = N$, then $|\mathcal{I}(G)| \leq |\mathcal{I}(K_{d,d})|^{N/d} = (2^{d+1} - 1)^{N/d}$

Extended to $|\mathrm{Hom}(G,H)|$ by Galvin & Tetali (2004) [G d-reg bipartite & H with loops but no multi-edges]

Extended to $|\mathcal{I}(G)|$ where G is d-regular graph by Zhao (2010) by a clever reduction.

Many open conjectures related to counting matchings, homomorphisms, etc.

Proof [$|\mathcal{I}(G)| \leq (2^{d+1}-1)^{N/d}$ where G is d-reg. bipartite on $2N$ vert]

Let I be an element of $\mathcal{I}(G)$ picked uniformly at random.
We represent I as $(X_v : v \in A \cup B)$ where $X_v = \begin{cases} 1 & \text{if } v \in I \\ 0 & \text{if } v \notin I \end{cases}$

$$\therefore \log(|\mathcal{I}(G)|) = H(I) = H(X_v : v \in A \cup B) = H(\underbrace{X_A}_{(X_v : v\in A)}, \overbrace{X_B}^{(X_v : v \in B)})$$

By Chain rule, $H(I) = H(X_B) + H(X_A \mid X_B)$

Proof $\left[\,|\mathcal{I}(G)| \leq (2^{d+1}-1)^{N/d}\right.$ where G is d-reg. bipartite on $2N$ vert$\left.\right]$

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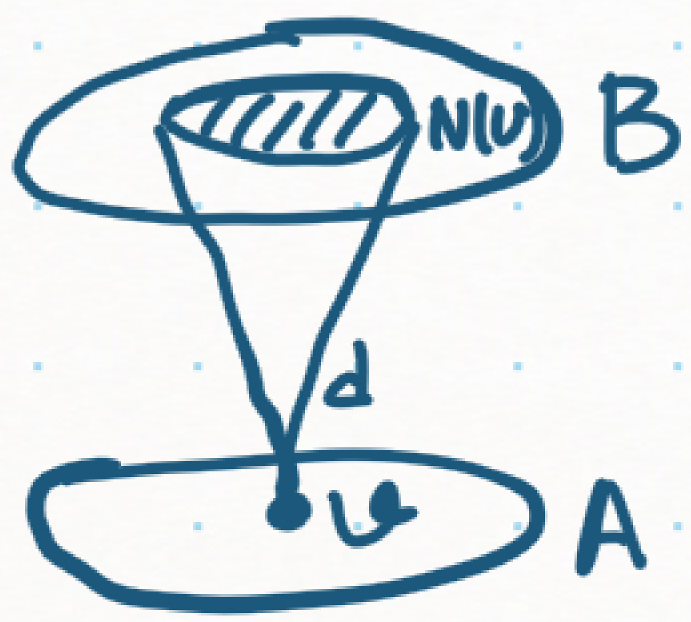

$\mathcal{A} = \{N(v) : v \in A\}$

Each $v \in B$ occurs in at least d neighborhoods of vertices in A.

$$\therefore H(X_B) \leq \frac{1}{d} \sum_{v \in A} H(X_{N(v)})$$

, by Shearer's lemma for projecting X_B onto $\mathcal{A}$ neighborhoods of $v \in A$ in B

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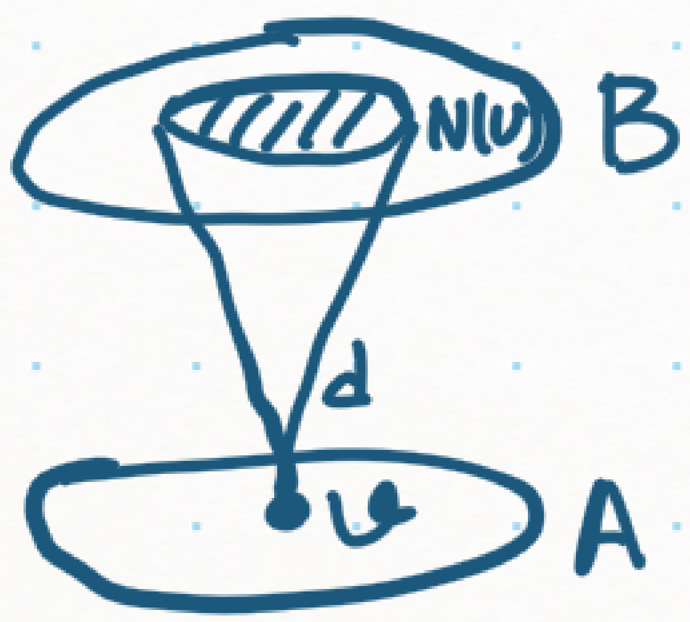

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For $H(X_A | X_B) \leq \sum_{v\in A} H(X_v | X_B)$, by subadditivity for conditional Entropy.

$$\leq \sum_{v\in A} H(X_v | X_{N(v)})$$
, by dropping conditioning since $N(v) \subseteq B$

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By Chain rule, $H(I) = H(X_B) + H(X_A | X_B)$

$$\leq \frac{1}{d} \sum_{v \in A} H(X_{N(v)}) + \sum_{v \in A} H(X_v | X_{N(v)})$$

$$\leq \frac{1}{d} \sum_{v \in A} \left(\underline{H(X_{N(v)})} + d\, \underline{H(X_v | X_{N(v)})} \right)$$

Global quantity has been reduced to local quantities in the neighborhoods.

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$$\leq \frac{1}{d} \sum_{v\in A} \left(H(X_{N(v)}) + d\, H(X_v \mid X_{N(v)}) \right)$$

For each $v \in A$, Let $\eta_v = \begin{cases} 0 & \text{if } X_{N(v)} = (0,0,\dots,0) \\ 1 & \text{otherwise} \end{cases}$ $\longleftarrow$ we can't pick v for I since one of its neighbors has been picked.

Let $p = \mathbb{P}[\eta_v = 0]$ then $H(X_v \mid X_{N(v)}) \leq H(X_v \mid \eta_v)$

we don't know p

because if we know $X_{N(v)}$ then we know η_v

$= H[X_v \mid \eta_v = 0]\, p + H(X_v \mid \eta_v = 1)(1-p)$

$\leq (\log 2)\, p + (\log 1)(1-p)$, by range of values

$= p$

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Let $p = \mathbb{P}[\eta_v = 0]$.

$$H(X_v | X_{N(v)}) \le p.$$

Since η_v is determined by $X_{N(v)}$, $H(X_{N(v)}) = H(X_{N(v)}, \eta_v)$

$$= H(\eta_v) + H(X_{N(v)} | \eta_v)$$

$$\le \underbrace{H(p)}_{\text{Binary entropy}} + p \log 1 + (1-p) \log(2^d - 1)$$

range of values

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$$\leq \frac{1}{d} \sum_{v \in A} \Big(H(X_{N(v)}) + d\, H(X_v | X_{N(v)}) \Big)$$

$$\leq \sum_{v\in A} \Big[p + \frac{1}{d}\big(H(p) + (1-p)\log(2^d - 1)\big)\Big]$$

$$= N\Big(p + \frac{1}{d}\big(H(p) + (1-p)\log(2^d-1)\big)\Big)$$

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$$= N \Big(\underbrace{p + \frac{1}{d}\left(H(p) + (1-p)\log(2^d-1)\right)}_{= f(p)} \Big)$$

Max $f(p)$ over all possible $p \in [0,1]$ gives $\frac{1}{d}\log(2^{d+1}-1)$ when $p = 2^d / (2^{d+1} - 1)$

$$\le N \left(\frac{1}{d} \log(2^{d+1} - 1)\right)$$

∎