Math 554

Hemanshu Kaul

kaul@iit.edu

A subgraph with specified vertex degrees

Given a graph G on n vertices, we want to find a subgraph H whose vertices have degrees in a subset of $\{0,1,2,\dots,n-1\}$.

For each $v \in V(G)$, a bad set $B(v) \subseteq \{0,1,2,\dots,d_G(v)\}$ is specified and we want H s.t. $d_H(v) \notin B(v)$.

Bad set of degrees

HW problem If $\sum_{v \in V(G)} |B(v)| < |E(G)|$ then we can do it.

(and $0 \notin B(v)\ \forall v$)

A subgraph with specified vertex degrees

Theorem [Shirazi- Verstraete 2008]

For each $v \in V(G)$, specify $B(v) \subseteq \{0,1,2,\dots,d_G(v)\}$.
If $|B(v)| \le \lfloor \frac{d_G(v)}{2} \rfloor$ $\forall v \in V(G)$, then G has a subgraph H with $d_H(v) \notin B(v)$ $\forall v$.

Sharp: Conclusion fails if even one $B(v)$ is too large.

Let $G = K_{2r,2r}$ with partite sets X, Y

Let $B(x) = \{0,\dots,r-1\}$ $\forall x \in X$
$B(y) = \{r+1,\dots,2r\}$ $\forall y \in Y$ $\Rightarrow |B(v)| = \frac{d(v)}{2}$ $\forall v$

A good subgraph must be r-regular.
But as soon as we include r in any one $B(v)$, then $|B(v)| = \frac{d(v)}{2}+1$ and there is no good subgraph H!

A subgraph with specified vertex degrees

Theorem [Shirazi- Verstraete 2008]

For each $v \in V(G)$, specify $B(v) \subseteq \{0, 1, 2, \dots, d_G(v)\}$.
If $|B(v)| \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor \ \forall v \in V(G)$, then G has a subgraph H with $d_H(v) \notin B(v) \ \forall v$.

Proof

Similar idea as Alon-Friedland-Kalai proof.

Design a polynomial $f(x)$ s.t. $f(x) \neq 0$ with $x_e \in \{0,1\}$ gives us a subgraph H corresponding to $x_e = 1$ with the required degrees.

A subgraph with specified vertex degrees

Theorem [Shirazi- Verstraete 2008]

For each $v \in V(G)$, specify $B(v) \subseteq \{0,1,2,\ldots,d_G(v)\}$.
If $|B(v)| \le \left\lfloor \frac{d_G(v)}{2} \right\rfloor$ $\forall v \in V(G)$, then G has a subgraph H with $d_H(v) \notin B(v)$ $\forall v$.

Proof: For each edge e, let $x_e \in \{0,1\}$ be a variable.
So, $x \in \{0,1\}^m$, where $m = |E(G)|$.

Let
$$f(x) = \prod_{v \in V(G)} \prod_{d \in B(v)} \left(\sum_{e \in \Gamma(v)} x_e - d \right)$$

A subgraph with specified vertex degrees

Theorem [Shirazi- Verstraete 2008]

For each $v \in V(G)$, specify $B(v) \subseteq \{0,1,2,\dots,d_G(v)\}$.
If $|B(v)| \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor$ $\forall v \in V(G)$, then G has a subgraph H with
$d_H(v) \notin B(v)$ $\forall v$.

Proof For each edge e, let $x_e \in \{0,1\}$ be a variable.
So, $x \in \{0,1\}^m$, where $m = |E(G)|$.

Let $$f(x) = \prod_{v \in V(G)} \prod_{d \in B(v)} \Big(\overbrace{\sum_{e \in \Gamma(v)} x_e}^{\text{degree of } v \text{ in } H} - d \Big)$$

$\Gamma(v)$ = set of edges incident to v

$\left[\left(\sum_{\Gamma(v)} x_e - d \right) = 0 \Leftrightarrow d_H(v) \in B(v) \right] \Rightarrow$ we want $x \in \{0,1\}^m$ s.t. $f(x) \neq 0$

A subgraph with specified vertex degrees

Theorem [Shirazi- Verstraete 2008]

For each $v \in V(G)$, specify $B(v) \subseteq \{0,1,2,\ldots,d_G(v)\}$.
If $|B(v)| \leq \left\lfloor \frac{d_G(v)}{2} \right\rfloor$ $\forall v \in V(G)$, then G has a subgraph H with $d_H(v) \notin B(v)$ $\forall v$.

Proof For each edge e, let $x_e \in \{0,1\}$ be a variable.
So, $x \in \{0,1\}^m$, where $m = |E(G)|$.

Let $f(x) = \prod_{v \in V(G)} \prod_{d \in B(v)} \left(\sum_{e \in \Gamma(v)} x_e - d \right)$, $\deg(f) \leq \sum_{v \in V(G)} |B(v)|$

To apply CN, we need a multilinear monomial of degree $\sum |B(v)|$ with non-zero coefficient.

Monomials in $f(x)$ arise by choosing, for each $d \in B(v)$, an edge incident to that vertex. Caution: should not pick an edge twice, once for each endpoint

To avoid picking an edge twice, we orient edges of G and pick for the monomial at v only the variables x_e s.t. v is the tail of e in the orientation: $\underset{v}{\bullet}\xrightarrow[e]{}\bullet$

If the orientation has at least $\lfloor \frac{d(v)}{2} \rfloor$ edges leaving each v, then there are enough edges to choose distinct ones for each $d \in B(v)$, since $|B(v)| \leq \lfloor \frac{d(v)}{2} \rfloor$.

<u>Want</u>: Orientation of G s.t. $d^+(v) \geq \lfloor \frac{d(v)}{2} \rfloor$

To avoid picking an edge twice, we orient edges of G and pick for the monomial at v only the variables x_e s.t. v is the tail of e in the orientation: $v \xrightarrow{e} \bullet$

If the orientation has at least $\lfloor \frac{d(v)}{2} \rfloor$ edges leaving each v, then there are enough edges to choose distinct ones for each $d \in B(v)$, since $|B(v)| \leq \lfloor \frac{d(v)}{2} \rfloor$.

Want: Orientation of G s.t. $d^+(v) \geq \lfloor \frac{d(v)}{2} \rfloor$

Add a vertex w adjacent to all odd degree vertices in G, forming G'.
Now, all degrees in G' are even & there exists an Eulerian circuit.
Orient the edges in G' according to this Eulerian circuit, giving a balanced orientation $d^+_{G'}(v) = d^-_{G'}(v) = \frac{d_{G'}(v)}{2}$
Ignore w to get the required orientation in G.

A subgraph with specified vertex degrees

Theorem [Shirazi- Verstraete 2008]

For each $v \in V(G)$, specify $B(v) \subseteq \{0,1,2,\dots,d_G(v)\}$.
If $|B(v)| \le \left\lfloor \frac{d_G(v)}{2} \right\rfloor$ $\forall v \in V(G)$, then G has a subgraph H with $d_H(v) \notin B(v)$ $\forall v$.

Proof For each edge e, let $x_e \in \{0,1\}$ be a variable.
So, $x \in \{0,1\}^m$, where $m = |E(G)|$.

Let $f(x) = \prod_{v \in V(G)} \prod_{d \in B(v)} \left(\sum_{e \in \Gamma(v)} x_e - d \right)$, $\deg(f) \le \sum_{v \in V(G)} |B(v)|$

Based on the orientation, we can pick a multilinear monomial of degree $\sum_v |B(v)|$ with positive coefficient. (since each choice of x_e from $e \in \Gamma(v)$, for each $d \in B(v)$ has positive contribution).

$\therefore$ by CN, $\exists\, x \in \{0,1\}^m$ s.t. $f(x) \ne 0$, so each $\left(\sum x_e - d\right) \ne 0$
i.e., $d_H(v) \notin B(v)$ $\forall v$ for H corresponding to $x_e = 1$. ∎

We know that a nonzero polynomial of degree d in a single variable has at most d zeros (roots).

Lemma 0 Let $f \in \mathbb{F}[x_1]$ s.t. $f(x_1) = \sum_{i=0}^{d} c_k x_1^k$ and f has at least $d+1$ roots, then $c_1 = c_2 = \cdots = c_d = 0$, i.e., $f \equiv 0$.

How can we generalize this to n variables?

Lemma 1 Let $f \in \mathbb{F}[x_1, \ldots, x_n]$. For each i, the degree of f as a polynomial in x_i be at most d_i, and let S_i be a set of d_i+1 distinct values in $\mathbb{F}$.
If $f(x_1, \ldots, x_n) = 0$ for $(x_1, \ldots, x_n) \in \prod_{i=1}^{n} S_i$, then $f \equiv 0$.

Combinatorial Nullstellensatz [Alon 1999, known since 1980s]
If $\prod_{i=0}^{n} x_i^{t_i}$ is a monomial with non-zero coefficient in $f \in \mathbb{F}[x_1, \ldots, x_n]$ where f has degree $\sum_{i=1}^{n} t_i$, and $S_1, \ldots, S_n \subseteq \mathbb{F}$ with $|S_i| > t_i$, $\forall i$, then $f(x) \neq 0$ for some $x \in \prod_{i=1}^{n} S_i$

Proof of CN

<u>Idea</u> Change f into another polyn $\hat{f}$ that agrees with f over $\prod_{i=1}^{n} S_i$, but has degree at most t_i as polyn in x_i $\forall i$

Then Lemma 1 implies $\hat{f}(x) \neq 0$ for some $x \in \prod_{i=1}^{n} S_i$

As we have $f(x) = \hat{f}(x)$ over $x \in \prod S_i$, we are done.

Proof of CN

Assume $|S_i| = t_i + 1 \ \forall i$ (If Thm is true for smaller S_i then....)

For each i, define $g_i(x_i) = \prod_{s \in S_i} (x_i - s) = x_i^{t_i+1} - \underbrace{h_i(x_i)}$

This is whatever is leftover after $x_i^{t_i+1}$
so, $\deg(h_i(x_i)) \le t_i$

note g_i has deg t_i+1 in x_i
and deg 0 in all other x_j

By definition, $g_i(x) = 0$ for $x \in \prod S_i$, since $x_i \in S_i$ in that case.

$\therefore$ $\underline{x_i^{t_i+1} = h_i(x_i) \ \forall x \in \prod S_i}$.

So, we replace each appearance of $x_i^{t_i+1}$ in $f(x)$ by $h_i(x_i)$, thus reducing $f(x)$ to $\hat{f}(x)$ with $\deg(\hat{f}) \le t_i$ in each x_i but $f(x) = \hat{f}(x)$ over $x \in \prod S_i$.

$[\prod x_i^{t_i}]\hat{f}(x) \neq 0$ since this monomial is the same in both f & $\hat{f}$ and we did not introduce any terms that could cancel it.

Apply Lemma 1 to $\hat{f}$, so $\hat{f}(x) \neq 0$ for some $x \in \prod S_i$, & so $f(x) \neq 0$ also. ∎

Coloring with Combinatorial Nullstellensatz

we know

$$\chi(G) \le \chi_\ell(G) \le \overbrace{\underbrace{\mathrm{Col}(G)}_{1+\underbrace{\text{degeneracy}(G)}_{\text{min } k \text{ s.t. every subgraph of } G \text{ has vertex of degree } \le k}}}^{\text{Greedy coloring bound}} \le \Delta(G)+1$$

or, order vertices in such a way that each v has at most k neighbors occur before it.

Coloring with Combinatorial Nullstellensatz

we know $\chi(G) \leq \chi_e(G) \leq \overbrace{\underbrace{Col(G)}_{1+\underbrace{\text{degeneracy}(G)}_{\substack{\text{min } k \text{ s.t. every subgraph} \\ \text{of } G \text{ has vertex of degree} \leq k}}}}^{\text{Greedy coloring bound}} \leq \Delta(G)+1$

<u>or</u>, order vertices in such a way that each v has at most k neighbors occur before it.

$$\chi(G) \leq \chi_e(G) \leq \underbrace{AT(G)}_{\text{Alon-Tarsi number of } G} \leq Col(G) \leq \Delta(G)+1$$

Coloring with Combinatorial Nullstellensatz

Let G be a graph with $V(G) = \{v_1, \ldots, v_n\}$.
The graph polynomial P_G of G is defined as

$$P_G(x_1, \ldots, x_n) = \prod_{\substack{i<j \text{ s.t.} \\ v_iv_j \in E(G)}} (x_i - x_j)$$

Note: one linear factor for each edge v_iv_j

Observations

- $P_G(x_1, \ldots, x_n) = 0 \iff$ assigning x_i to v_i $\forall i$ produces a monochromatic edge

- Given a G with list assignment $L(v)$ $\forall v \in V(G)$, setting $S_i = L(v_i)$ & apply CN successfully would give us a proper L-coloring of G.

[First defined by Petersen 1891, who introduced graphs to study such polynomials]

Coloring with Combinatorial Nullstellensatz

Let G be a graph with $V(G)=\{v_1,\dots,v_n\}$.
The graph polynomial P_G of G is defined as

$$P_G(x_1,\dots,x_n) = \prod_{\substack{i<j \text{ s.t.} \\ v_iv_j \in E(G)}} (x_i - x_j)$$

Note one linear factor for each edge v_iv_j

In order to apply CN to P_G, we need an understanding of coefficients of P_G.

Note that if we change the linear order $v_1,\dots,v_n$, we will only change the sign of P_G by ± 1.

In general, we could define P_G using any orientation D of G.

$$P_D(\vec{x}) = \prod_{(u,v)\in E(D)} (x_u - x_v).$$

We are only interested in coefficient of a specific monomial being non-zero

Lemma For an orientation D of G with $d_i = d_D^+(v_i)$ $\forall i$,

$$\left| \left[\prod x_i^{d_i} \right] P_G \right| = \mathrm{diff}(D),$$

where $\mathrm{diff}(D) = \left| \#\text{Even Circulations in } D - \#\text{ Odd Circulations in } D \right|$

Defn D is circulation if $d_D^+(v) = d_D^-(v)$ $\forall v \in V(D)$

(i.e., each component is Eulerian)

Even circulation is a circulation in D with even # edges.

Odd ———————— " ———————— odd # edges.

Theorem [Alon, Tarsi 1992] Let G be a graph with list assignment L, s.t. $|L(v)| \geq 1 + d_D^+(v)$ for an orientation D of G.

If $\mathrm{diff}(D) \neq 0$ then G is L-colorable.

Given G, find D s.t. $\mathrm{diff}(D) \neq 0$!

Lemma For an orientation D of G with $d_i = d_D^+(v_i)\ \forall i$,

$$\left| \left[\prod x_i^{d_i}\right] P_G \right| = \mathrm{diff}(D),$$

where $\mathrm{diff}(D) = \left| \#\underset{\text{in } D}{\text{Even Circulations}} - \#\underset{\text{in } D}{\text{Odd Circulations}} \right|$

Theorem [Alon Tarsi 1992] Let G be a graph with list assignment L. s.t. $|L(v)| \geq 1 + d_D^+(v)\ \forall v$ for some orientation D of G. If $\mathrm{diff}(D) \neq 0$ then G is L-colorable.

Proof By Lemma $\left| \left[\prod x_i^{d_i}\right] P_G \right| = \mathrm{diff}(D) \neq 0$

so by CN, $\exists x \in \prod S_i$, i.e., $x_i \in L(v_i)\ \forall i$ such that

$P_G(x) \neq 0$, when each $|S_i| = |L(v_i)| \geq d_i + 1$.

$\therefore G$ is L-colorable.

Lemma For an orientation D of G with $d_i = d_D^+(v_i)$ $\forall i$,

$$\left| \left[\prod x_i^{d_i} \right] P_G \right| = \mathrm{diff}(D),$$

where $\mathrm{diff}(D) = \left| \#\text{Even Circulations in } D - \#\text{Odd Circulations in } D \right|$

Theorem [Alon Tarsi 1992] Let G be a graph with list assignment L. s.t. $|L(v)| \geq 1 + d_D^+(v)$ $\forall v$ for some orientation D of G.
If $\mathrm{diff}(D) \neq 0$ then G is L-colorable.

Definition Alon-Tarsi number of G,

$$AT(G) = \min \left\{ k : \begin{array}{l} G \text{ has an orientation } D \text{ with} \\ d_D^+(v) \leq k-1 \ \forall v \in V(G) \end{array} \right\}$$

Then, $\chi_\ell(G) \leq AT(G)$

Let $G = C_n$

$D =$ cyclic orientation of G

What circulations are contained in D?

Let $G = C_n$

D = cyclic orientation of G

What circulations are contained in D?

→ Empty circulation (no edges)

→ D itself

What is $\mathrm{diff}(D)$?

Let $G = C_n$

D = cyclic orientation of G

What circulations are contained in D?

→ Empty circulation (no edges)

→ D itself

What is $\mathrm{diff}(D)$?

If n is even then 2 even circulations & no odd circ.

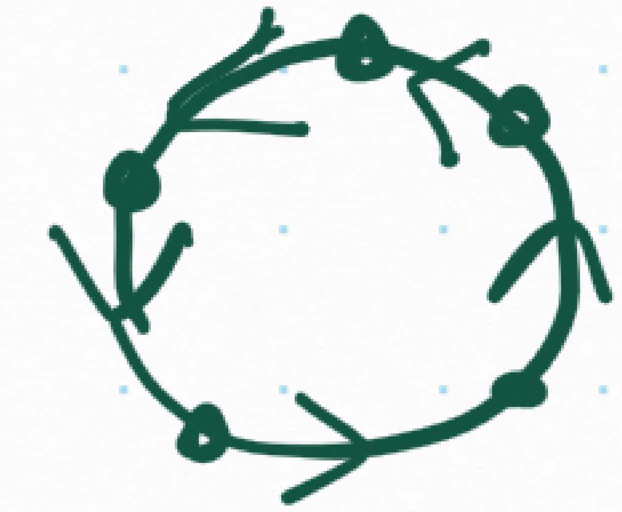
so, $\mathrm{diff}(D) = 2 - 0 = 2 \neq 0$

$\therefore \chi_e(C_n) \leq 2$ when n even

$\Delta^+(D) = 1$, so $AT(G) \leq 1 + 1 = 2$

Let $G = C_n$

D= cyclic orientation of G



What circulations are contained in D?

→ Empty circulation (no edges)

→ D itself

What is diff(D)?

If n is even then 2 even circulations & no odd circ.

so, $\mathrm{diff}(D) = 2 - 0 = 2 \neq 0$

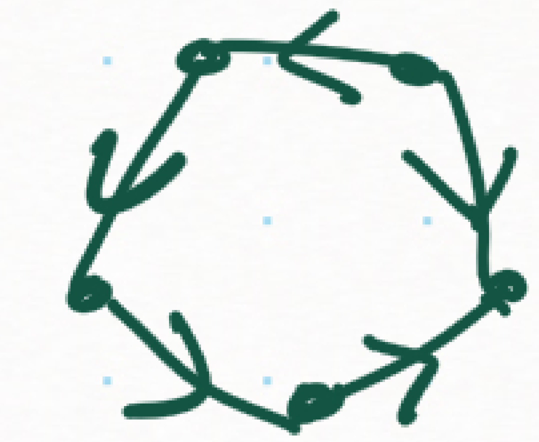
$\therefore \chi_e(C_n) \leq 2$ when n even

If n is odd Then $\mathrm{diff}(D) = 1 - 1 = 0$, useless.

not completely: $AT(C_n) > 2$ for n odd.

Let $G = C_n$

$D' =$ cyclic orientation with one edge reversed



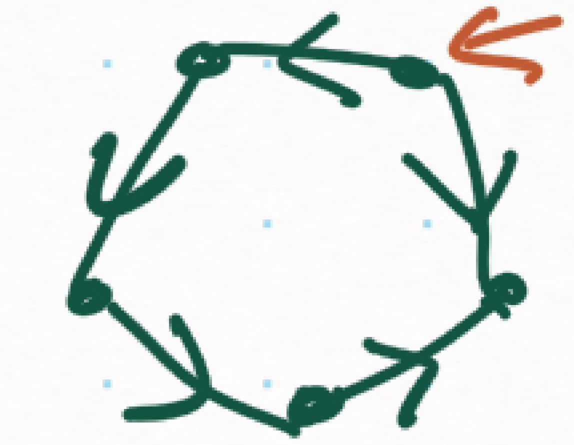
Only a single circulation, the empty one.

$\therefore \operatorname{diff}(D') = 1 - 0 = 1 \neq 0$

Thus, $\chi_e(C_n) \leq AT(C_n) \leq$

Let $G = C_n$

$D' =$ cyclic orientation with one edge reversed



Only a single circulation, the empty one.

$\therefore \operatorname{diff}(D') = 1 - 0 = 1 \neq 0$

Thus, $\chi_e(C_n) \leq AT(C_n) \leq 1 + \underbrace{2}_{\Delta^+(D)} = 3$ when n odd

$$\therefore \chi_e(C_n) \leq AT(C_n) \leq \begin{cases} 2 & \text{if } n \text{ even} \\ 3 & \text{if } n \text{ odd} \end{cases}$$

Observations

① Recall $\chi(G) \leq \chi_\ell(G) \leq AT(G) \leq Col(G) \leq \Delta(G)+1$

easy. By defn.

CN & Alon-Tarsi

?

easy
any greedy order of vertices works.

Let G be k-degenerate. Then $AT(G) \leq k+1$.

Proof Find an orientation of G using the "$Col(G)$ ordering" of vertices that has no directed cycles.
Then what is #Even Circ.? what is #Odd Circ.?

Observations

① Recall $\chi(G) \le \chi_\ell(G) \le AT(G) \le Col(G) \le \Delta(G)+1$

easy. By defn.

CN & Alon-Tarsi

?

easy
any greedy order of vertices works.

Let G be k-degenerate. Then $AT(G) \le k+1$.

Proof Find an orientation of G using the "$Col(G)$ ordering" of vertices that has no directed cycles.
Then what is #Even Circ.? what is #~~Odd~~ Circ.?

② Each of χ, χ_ℓ, Col, etc. has the property $f(H) \le f(G)$ for any subgraph H of G

What about AT?

$AT(H) \le AT(G)$ for any subgraph H of G.

Proof?

Lemma [Tarsi 1981]

Every graph G has an orientation s.t. $\Delta^+(D) \leq \rho(G)$

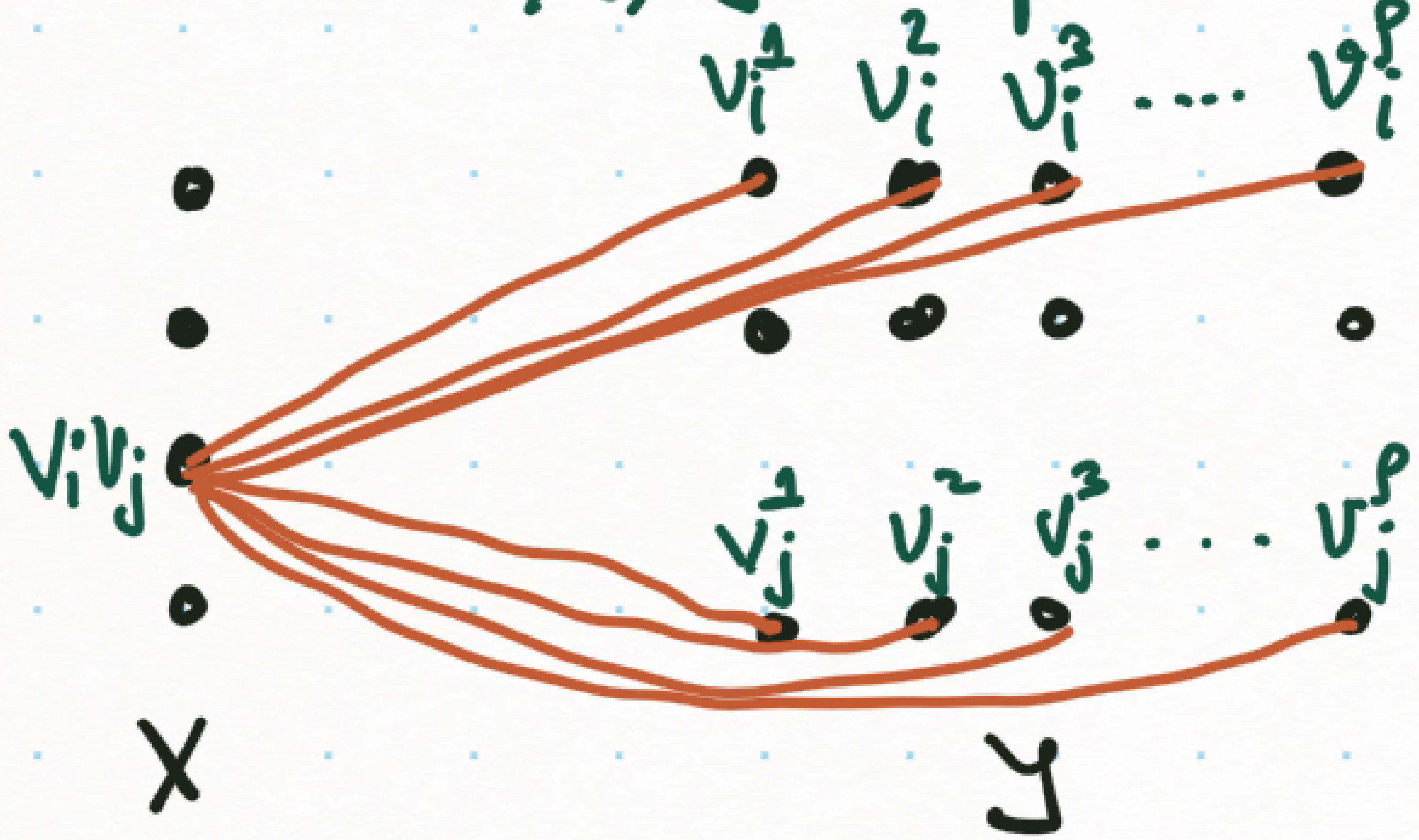
where $\rho(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|} \right\rceil$

Lemma [Tarsi 1981]

Every graph G has an orientation s.t. $\Delta^+(D) \leq \rho(G)$
where $\rho(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|} \right\rceil$

Proof Let $\rho = \rho(G)$

Form X,Y-bipartite graph from G as: $X = E(G)$, $Y = V(G) \times [\rho]$



each v_iv_j is adjacent to each copy of v_i an v_j.

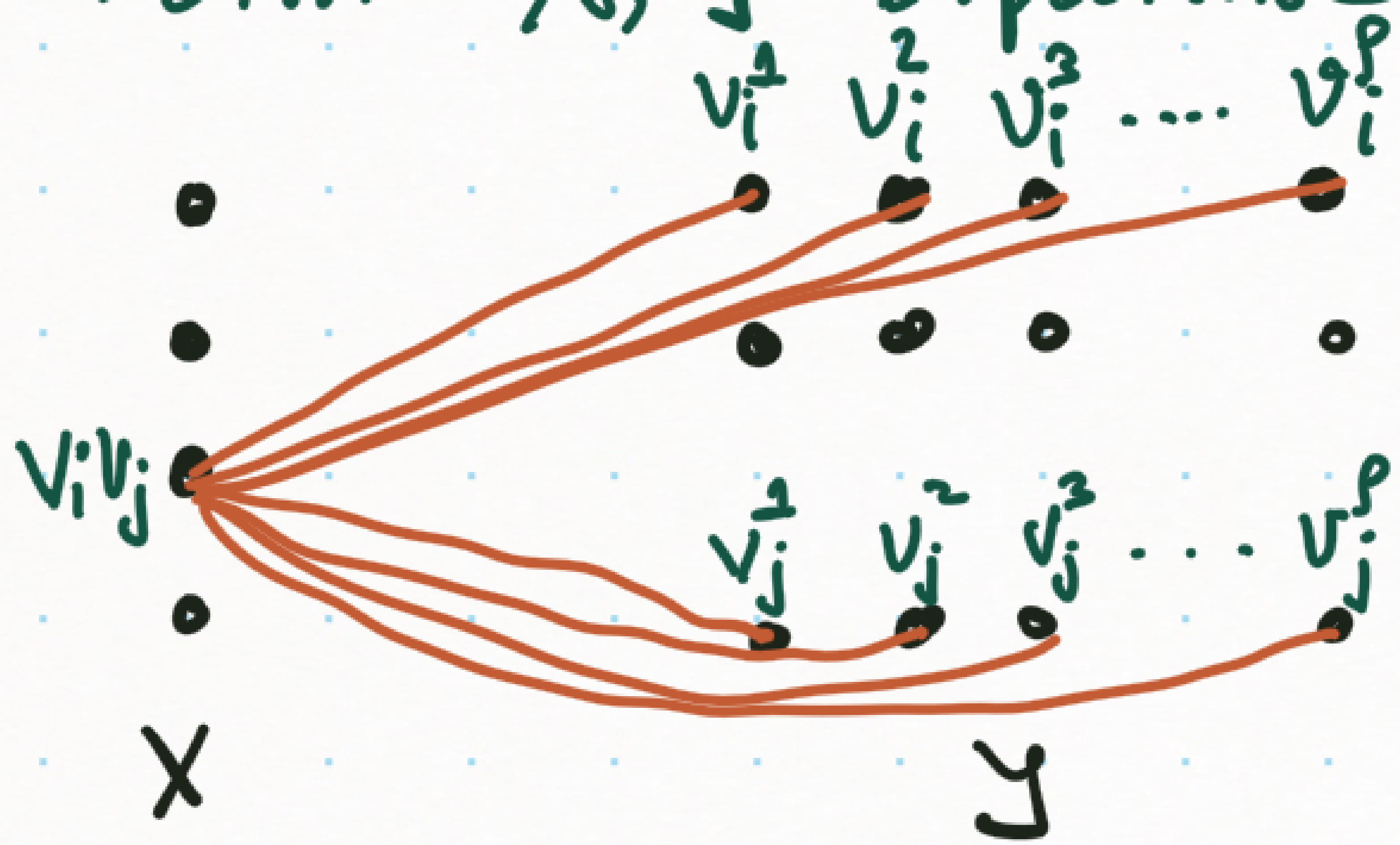
Claim $\exists$ matching saturating X

Lemma [Tarsi 1981]

Every graph G has an orientation s.t. $\Delta^+(D) \leq \rho(G)$
where $\rho(G) = \max\limits_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|} \right\rceil$

Proof Let $\rho = \rho(G)$

Form X, Y-bipartite graph from G as: $X = E(G)$, $Y = V(G) \times [\rho]$



each v_iv_j is adjacent to each copy of v_i or v_j.

Claim $\exists$ matching saturating X

Pf. Apply Hall's thm.

For $S \subseteq X$, $\exists H \subseteq G$ with $S = E(H)$ & $N(S) = V(H) \times [\rho]$

Then, $|N(S)| = |V(H)|\rho \geq |E(H)| = |S|$.

$\uparrow$ by defn. of ρ

Orient $v_i \to v_j$ in D when v_iv_j is matched to some copy of v_i, v_i^k.

This orientation D has $\Delta^+_D \leq \rho$ as needed. ■

Lemma [Tarsi 1981]

Every graph G has an orientation s.t. $\Delta^+(D) \le \rho(G)$

where $\rho(G) = \max\limits_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|} \right\rceil$

Cor If G is a bipartite graph, then $AT(G) = \rho(G)$

Proof $AT(G) \le \rho(G)$?

$AT(G) \ge \rho(G)$?

Cor [Alon Tarsi 1992] If G is bipartite planar then $\chi_\ell(G) \le 3$.

Proof Euler's formula: $n - m + f = 2$

Bipartite planar means each face has length ≥ 4, so $2m \ge 4f$

So by Euler $m \le 2n - 4$ for any such graph.

Since any $H \subseteq G$ is also bipartite planar, $\left\lceil \frac{|E(H)|}{|V(H)|} \right\rceil \le 2$. So, $\rho(G) \le 2$

$\therefore \exists$ orientation with $\Delta^+(D) \le 2$.

Every circulation D' in D decomposes into even cycles (since G is bipartite).

$\therefore |E(D')| =$ even. And no odd circul. Hence $\mathrm{diff}(D) \neq 0$. By A-Tarsi, $\chi_\ell(G) \le 3$.

Lemma [Tarsi 1981]

Every graph G has an orientation s.t. $\Delta^+(D) \leq \rho(G)$

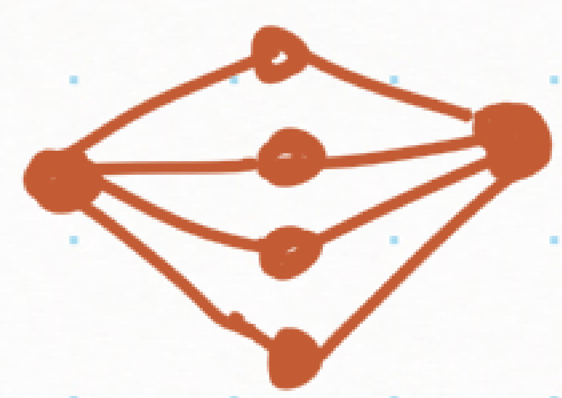
where $\rho(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|} \right\rceil$

Cor If G is a bipartite graph, then $AT(G) = \rho(G)$

Proof $AT(G) \leq \rho(G)$?

$AT(G) \geq \rho(G)$?

Cor [Alon Tarsi 1992] If G is bipartite planar then $\chi_\ell(G) \leq 3$.

Sharp! $K_{2,4}$  is bipartite planar and $\chi_\ell(K_{2,4}) > 2$.

Why?

In undergrad Graph Theory, we study Thomassen's proof that $\chi_\ell(G) \leq 5$ for any planar graph G.

Recently, Xuding Zhu extended this to show

$$AT(G) \leq 5 \text{ for any planar graph } G.$$

and more, Zhu & co-authors also showed

Every planar graph G contains a matching M s.t.

$$AT(G \setminus M) \leq 4.$$

(all these bounds are sharp)

Math 554

Hemanshu Kaul

kaul@iit.edu

Counting via Markov Chain Monte Carlo (MCMC)

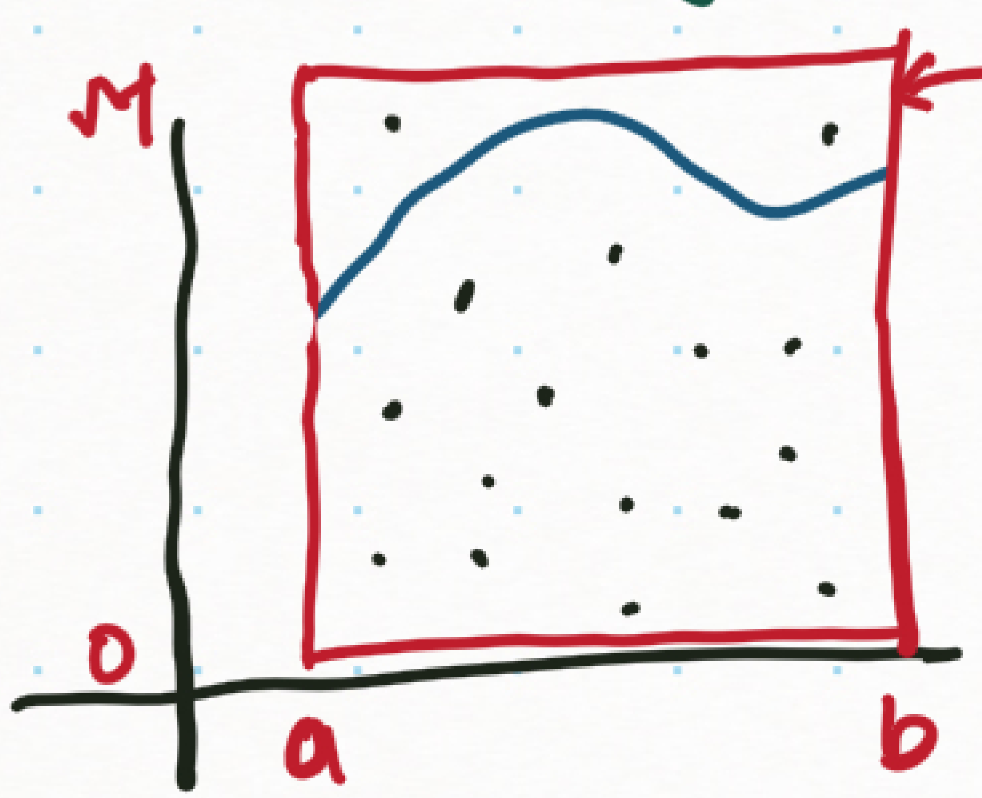
Classical Monte Carlo is a method for estimating quantities that are hard to compute.

$$z = \mathbb{E}[Z]$$

quantity we want to compute

random variable that samples underlying "objects" from a probability space Ω

As long as there is an efficient procedure for sampling from Ω we can take the mean of sufficiently large set of independent samples of Z, to get an approximation of z.



Bounding Box

Generate random points in BBox unit.

Then $\boxed{\text{area under the curve} \propto \left(\dfrac{\#\text{successful trials}}{\#\text{Total Trials}}\right)(\text{Area of BBox})}$

Counting via Markov Chain Monte Carlo (MCMC)

Classical Monte Carlo is a method for estimating quantities that are hard to compute.

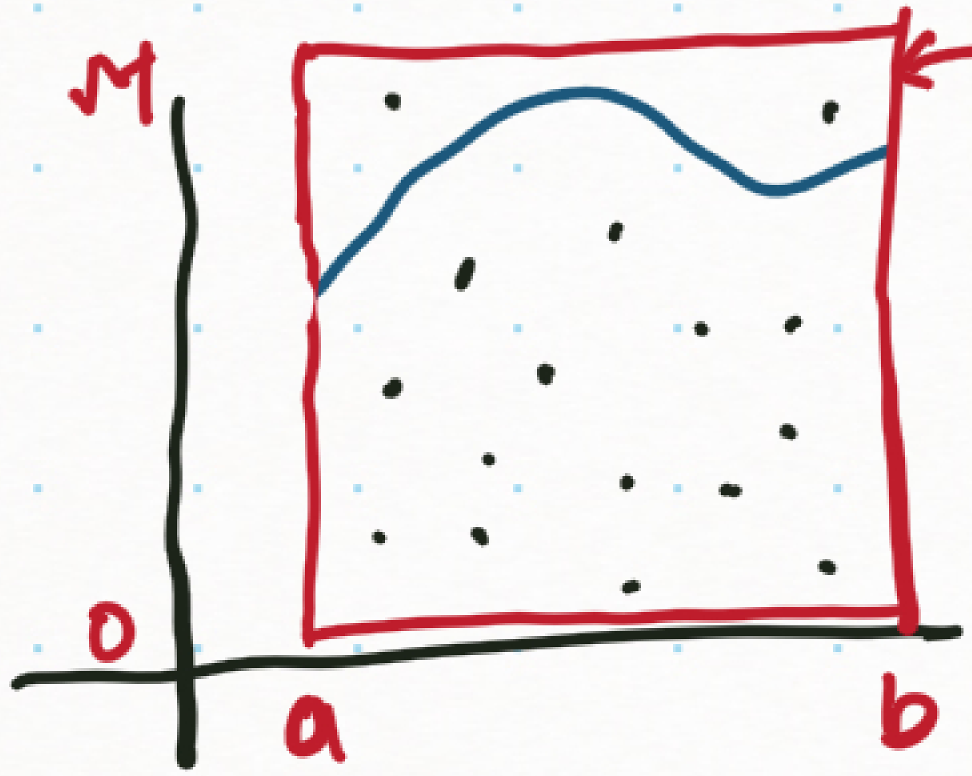
|S| where S is a collection of ind. sets of G, etc.

$$z = \mathbb{E}[Z]$$

quantity we want to compute

random variable that samples underlying "objects" from a probability space Ω

As long as there is an efficient procedure for sampling from Ω we can take the mean of sufficiently large set of independent samples of Z, to get an approximation of z.



Bounding Box

Generate random points in BBox unit.

Then $\boxed{\text{area under the curve} \propto \left(\frac{\#\text{successful trials}}{\#\text{Total Trials}}\right)(\text{Area of BBox})}$

But what if the "objects" to be sampled are not points but are combinatorial structures?

How can we sample?

Short review of Homogenous Discrete Time Markov Chains

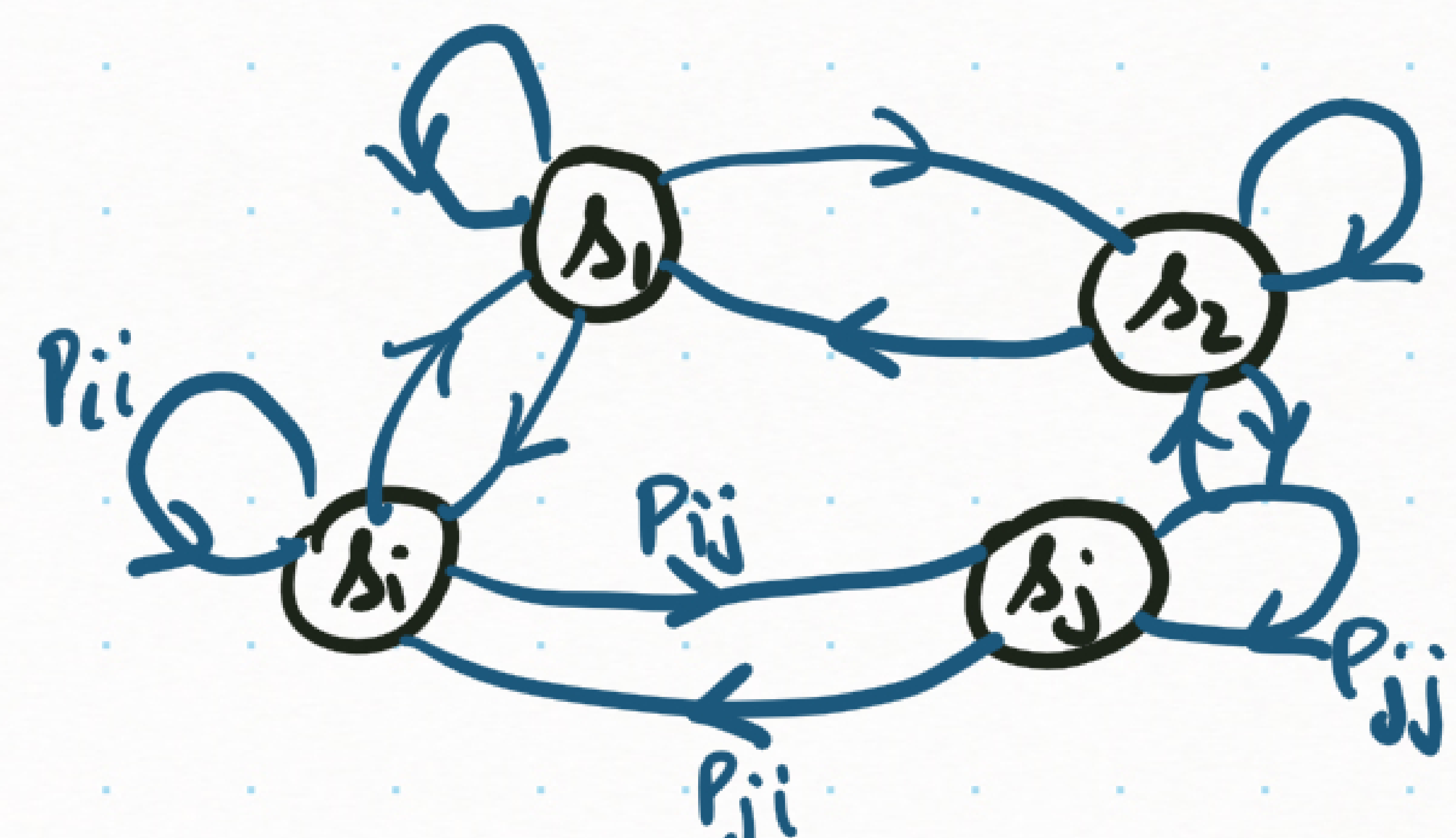
A random process $(X_0, X_1, \dots)$ on a finite space $S = \{s_1, \dots, s_k\}$ is said to be a <u>HDT Markov Chain</u> with transition matrix $P_{k \times k}$ matrix

if $\forall n, \forall i,j \in [k], \forall i_0, \dots, i_{n-1} \in [k]$

$$\mathbb{P}[X_{n+1} = s_j \mid X_0 = s_{i_0}, X_1 = s_{i_1}, \dots, X_{n-1} = s_{i_{n-1}}, X_n = s_i]$$

$$= \mathbb{P}[X_{n+1} = s_j \mid X_n = s_i] \leftarrow \text{Memorylessness property / Markov prop.}$$

$$= P_{ij} \leftarrow (i,j) \text{ entry of } P$$



<u>an edge wt.ed</u>
<u>digraph on S</u>
state space

- $P_{ij} \in [0,1]$
- $\sum_{j=1}^{k} P_{ij} = 1 \quad \forall i$

Short review of Homogenous Discrete Time Markov Chains

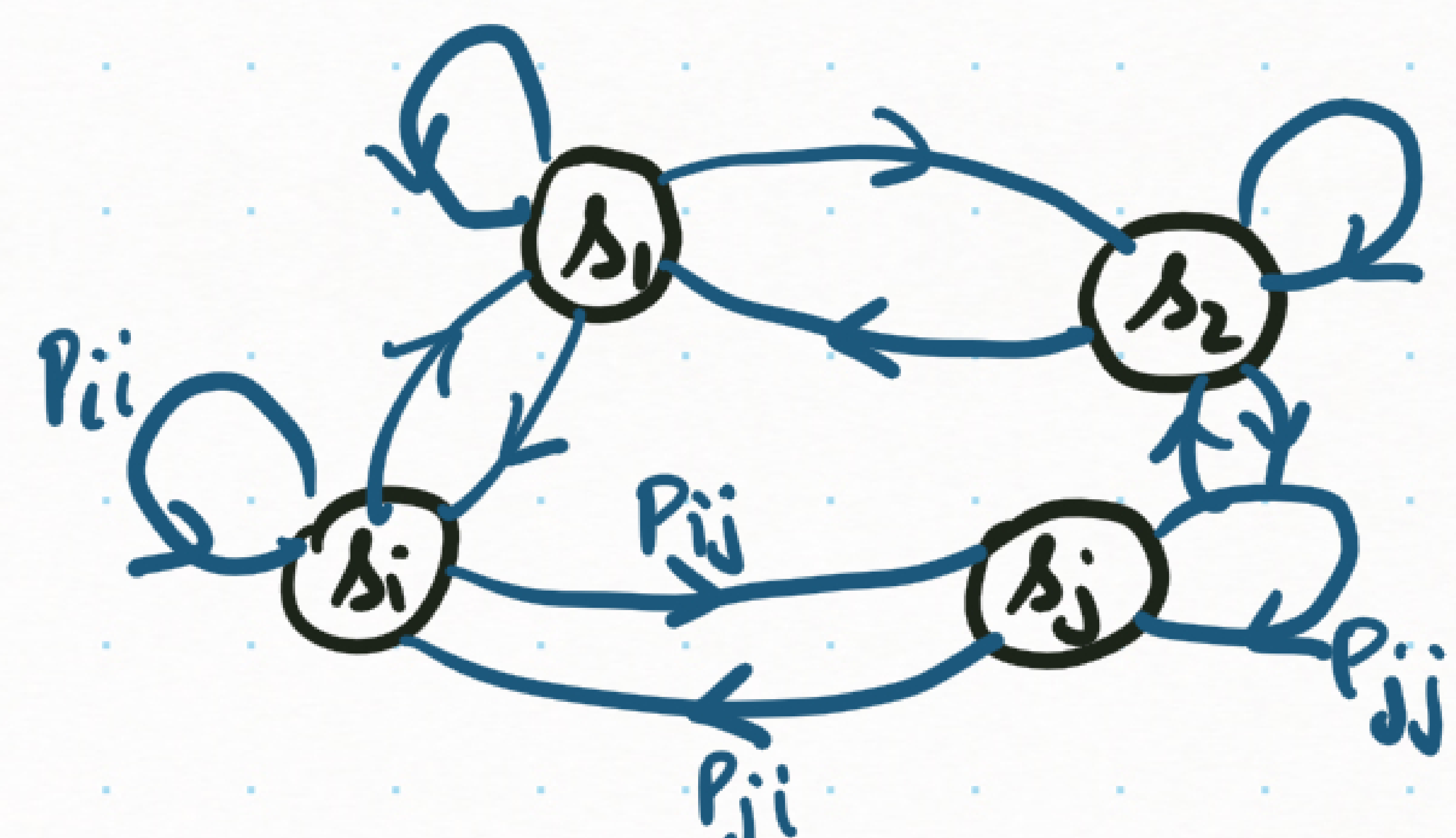
A random process $(X_0, X_1, \dots)$ on a finite space $S = \{s_1, \dots, s_k\}$ is said to be a HDT Markov Chain with transition matrix $P_{k\times k}$ matrix

if $\forall n$, $\forall i,j \in [k]$, $\forall i_0, \dots, i_{n-1} \in [k]$

$$\mathbb{P}[X_{n+1} = s_j \mid X_0 = s_{i_0}, X_1 = s_{i_1}, \dots, X_{n-1} = s_{i_{n-1}}, X_n = s_i]$$

$$= \mathbb{P}[X_{n+1} = s_j \mid X_n = s_i] \leftarrow \text{Memorylessness property / Markov prop.}$$

$$= P_{ij} \leftarrow (i,j) \text{ entry of } P$$



on edge wt.ed digraph on S

state space

- $P_{ij} \in [0,1]$
- $\sum_{j=1}^{k} P_{ij} = 1 \quad \forall i$
- Initial distribution $\mu^{(0)} = (\mu_1^{(0)}, \dots, \mu_k^{(0)}) = (\mathbb{P}[X_0 = s_1], \dots, \mathbb{P}[X_0 = s_k])$

  After n-steps $\mu^{(n)} = (\mu_1^{(n)}, \dots, \mu_k^{(n)}) = (\mathbb{P}[X_n = s_1], \dots, \mathbb{P}[X_n = s_k])$

  $= \mu^{(0)} P^n$

Two fundamental conditions on Markov Chains

① MC is called irreducible if the corresponding transition digraph is strongly connected. (can go from each state to another w. positive probab)

i.e., $\forall i,j \ \exists n \geq 0$ s.t. $\mathbb{P}[X_{m+n} = s_i \mid X_m = s_j] > 0$

s_i communicates with s_j, $s_i \leftrightarrow s_j$ if $\exists n \geq 0$ s.t. $[P^n]_{ij} > 0$

② Period $d(s_i)$ of state s_i is $\gcd\{n \geq 1 : [P^n]_{i,i} > 0\}$

If $d(s_i) = 1 \ \forall s_i$ then MC is called aperiodic.

Two fundamental conditions on Markov Chains

① MC is called <u>irreducible</u> if the corresponding transition digraph is strongly connected. (can go from each state to another w. positive probab.)

i.e., $\forall i,j \;\exists n \geq 0$ s.t. $\mathbb{P}[X_{m+n} = s_i \mid X_m = s_j] > 0$

s_i communicates with s_j, $s_i \leftrightarrow s_j$ if $\exists n \geq 0$ s.t. $[P^n]_{ij} > 0$

② <u>Period</u> $d(s_i)$ of state s_i is $\gcd\{n \geq 1 : [P^n]_{i,i} > 0\}$

If $d(s_i) = 1 \;\forall s_i$ then MC is called <u>aperiodic</u>.

<u>Theorem</u> Let (X_n) be an $\overbrace{\text{Ergodic}}^{\text{irreducible \& aperiodic}}$ MC with state space $S = \{s_1, \ldots, s_k\}$ and transition matrix P, then $\exists N < \infty$ s.t. $[P^n]_{ij} > 0 \;\forall i,j \in [k]$ & $\forall n \geq N$

How can we ensure our MC is ergodic?

$\rightarrow$ strong connectivity $\quad \rightarrow P_{ii} > 0$

What happens as we run a MC for a long time?

$\pi = (\pi_1, \dots, \pi_k)$ is s.t.b. a <u>stationary distribution</u> for the MC if

- $\pi_i \geq 0 \ \forall i$ and $\sum_{i=1}^{k} \pi_i = 1$
- $\underbrace{\pi P = \pi}_{\text{left eigenvector of } P \text{ w.r.t. eigenvalue } 1.}$, i.e., $\sum_{i=1}^{k} \pi_i P_{ij} = \pi_j \ \forall j$

Note that if $\mu^{(0)} = \pi$ then $\mu^{(1)} = \mu^{(0)} P = \pi P = \pi$ & so on $\mu^{(n)} = \pi$.

<u>Theorem</u> For any Ergodic MC, $\exists$ unique stationary distribution

What happens as we run a MC for a long time?

$\pi = (\pi_1, \dots, \pi_k)$ is s.t.b. a stationary distribution for the MC if

- $\pi_i \geq 0 \ \forall i$ and $\sum_{i=1}^{k} \pi_i = 1$
- $\underbrace{\pi P = \pi}_{\text{left eigenvector of } P \text{ w.r.t. eigenvalue } 1.}$, i.e., $\sum_{i=1}^{k} \pi_i P_{ij} = \pi_j \ \forall j$

Note that if $\mu^{(0)} = \pi$ then $\mu^{(1)} = \mu^{(0)} P = \pi P = \pi$ & so on $\mu^{(n)} = \pi$.

Theorem For any Ergodic MC, $\exists$ unique stationary distribution

Theorem For any Ergodic MC starting from an arbitrary initial distribution $\mu^{(0)}$, $\underbrace{\mu^{(n)} \longrightarrow \pi}_{\text{What does this mean? How is "close to } \pi \text{" defined}}$ as $n \to \infty$

Let $\nu = (\nu_1, \dots, \nu_k)$ & $\mu = (\mu_1, \dots, \mu_k)$ be two probability distributions on S then Total variation distance between ν and μ is:

$$\|\nu - \mu\|_{TV} = \frac{1}{2} \sum_{i=1}^{k} |\nu_i - \mu_i| = \max_{A \subseteq S} |\nu(A) - \mu(A)|$$

How to find the stationary distribution?

How to design a MC with appropriate stationary distribution?

A probability distribution π on S is said to be reversible for P

if $\forall\, i,j \in [k]$, $\pi_i P_{ij} = \pi_j P_{ji}$ $\longleftarrow$ at each step of MC $\mathbb{P}[i \to j] = \mathbb{P}[j \to i]$

MC is s.t.b. reversible if $\exists$ a reversible distribution for it.

Theorem If π is a reversible distribution for a MC, then it is also a stationary distribution for that MC.

$$\sum_{i=1}^{k} \pi_i P_{ij} = \sum_{i=1}^{k} \pi_j P_{ji} = \pi_j \sum_{i=1}^{k} P_{ji} = \pi_j \text{ , i.e., } \pi P = \pi$$

If we design an Ergodic MC with a reversible distribution then we know the unique stationary distribution of the MC.

Random walk on a finite graph

$S = V(G) = \{v_1, \dots, v_k\}$

Move from v_i to one of its neighbors uniformly at random $\left(d_i = \deg(v_i)\right)$

$$P_{ij} = \begin{cases} 1/d_i & \text{if } j \in N(v_i) \\ 0 & \text{otherwise} \end{cases}$$

- P is irreducible $\Leftrightarrow$
- P is aperiodic $\Leftrightarrow$

Random walk on a finite graph

$S = V(G) = \{v_1, \dots, v_k\}$

Move from v_i to one of its neighbors uniformly at random $(d_i = \deg(v_i))$

$$P_{ij} = \begin{cases} 1/d_i & \text{if } j \in N(v_i) \\ 0 & \text{otherwise} \end{cases}$$

- P is irreducible $\Leftrightarrow$ G is connected
- P is aperiodic $\Leftrightarrow$ G is not bipartite
- P is reversible with distribution $\pi_i = d_i / 2|E(G)|$

Pf. $v_i v_j \notin E(G) \Rightarrow P_{ij} = P_{ji} = 0$

$v_i v_j \in E(G) \Rightarrow \pi_i P_{ij} = \dfrac{d_i}{2|E(G)|} \dfrac{1}{d_i} = 1/2|E(G)| = \dfrac{d_j}{2|E(G)|} \dfrac{1}{d_j} = \pi_j P_{ji}$

For a non-bipartite connected graph G, this MC has a unique stationary distribution given by $\pi_i = \dfrac{d_i}{2|E(G)|} = \dfrac{d_i}{\sum_{j=1}^{k} d_j}$

What does it mean to run this MC on such graph?