Math 554

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Probabilistic Method To prove an object exists, define an appropriate probability space where in a random construction of the object works with positive probability.

Theorem Every graph G contains a bipartite subgraph with at least $|E(G)|/2$ edges.

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Theorem Every graph G contains a bipartite subgraph with at least $|E(G)|/2$ edges.

Proof Randomly color each vertex of G with 0 or 1 independently u.a.r. ⟵ what does this mean here?

Let E' = set of edges with one endpt. 0 and other 1.

Then $(V(G), E')$ is a bipartite subgraph of G.

Each edge belongs to E' with probability ½

$\therefore \mathbb{E}[|E'|] = \frac{1}{2}|E(G)|$ by lin. of exp. Hence $\exists$ a coloring with $|E'| \geq \frac{1}{2}|E(G)|$ as needed

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Proof Randomly color each vertex of G with 0 or 1 ind. w. prob. $1/2$

Let $X_e = \begin{cases} 1 & \text{if endpoints of } e \text{ have different colors} \\ 0 & \text{otherwise} \end{cases}$ Indicator r.v. for "good" edges

Then $X = \sum_{e \in E(G)} X_e$ counts the number of edges in the bipartite subgraph.

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$$\mathbb{E}[X] = \sum_e \mathbb{E}[X_e] = \sum_e \mathbb{P}[X_e = 1] = \sum_e \left(\tfrac{1}{4} + \tfrac{1}{4}\right) = \sum_e \tfrac{1}{2} = \tfrac{1}{2}|E(G)|$$

$\therefore \exists$ coloring with $X \geq \frac{1}{2}|E(G)|$ by pigeonhole property.

<u>Recall</u> In a group of 6 or more people,
there are 3 mutual acquaintances or
3 mutual strangers.

i.e., any graph on 6 or more vertices contains K_3 or $\overline{K_3}$

i.e., any 2-coloring of $E(K_6)$ contains a monochromatic K_3.

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<u>Defn</u> $R(k,l) =$ smallest n s.t. in every red-blue edge coloring of K_n there is a red K_k or blue K_l

$$R(k,l) = n \quad \begin{cases} \rightarrow R(k,l) \le n \\ \rightarrow R(k,l) > n-1 \end{cases}$$

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Defn $R(k,\ell) =$ smallest n s.t. in every red-blue edge coloring of K_n there is a red K_k or blue K_ℓ

$R(k,\ell) = n$ → $R(k,\ell) \le n$ all colorings are "good"
→ $R(k,\ell) > n-1$ $\exists$ "bad" coloring

e.g. $R(3,3) = 6$ ← Do it again!

Ramsey (1929) proved $R(k,\ell)$ exists and is finite.
↖ more general than that.

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Let $A_S \equiv$ event that S induces a monochromatic K_k, for $S \in \binom{V(K_n)}{k}$

family of all k-subsets of $V(K_n)$

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$$\mathbb{P}[A_S] = \frac{1}{2}^{\binom{k}{2}} + \frac{1}{2}^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

$$\mathbb{P}[\exists \text{ monochromatic } K_k] = \mathbb{P}\Big[\bigcup_{S \in \binom{V(G)}{k}} A_S\Big] \leq \sum \mathbb{P}[A_S]$$
$$= \binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

$\therefore$ with positive probability $\exists$ random coloring with no monochromatic K_k

Explicit bound on $R(k,k)$?

$$\binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

optimize n using estimates on the Binomial coeficient

i.e., $2\binom{n}{k} < 2^{\frac{k(k-1)}{2}}$

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Since $\binom{n}{k} < (ne/k)^k$, it suffices to have

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i.e., $$\frac{2ne}{k} < 2^{(k-1)/2}$$

i.e, $$n < \left(\frac{1}{e(\sqrt{2})^3}\right) k\,(\sqrt{2})^k$$

So, we get

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So, we get

$$\left(\frac{1}{e(\sqrt{2})^3}\right) k (\sqrt{2})^k < \overbrace{\left(\frac{1}{e\sqrt{2}} + o(1)\right) k (\sqrt{2})^k}^{\text{using better estimates}} < R(k,k)$$

while best known upper bd. is $(4+o(1))^k$

Some estimates / approximations

- Stirling Formula $\binom{n}{k} = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{n}{k(n-k)}} \left(\frac{n}{k}\right)^k \left(\frac{n}{n-k}\right)^{n-k} R_{n,k}$

  where $0.881 \leq \exp\left(-\frac{1}{12k} - \frac{1}{12(n-k)} + \frac{1}{12n+1}\right) < R_{n,k} <$

  $< \exp\left(-\frac{1}{12k+1} - \frac{1}{12(n-k)+1} + \frac{1}{12n}\right) < 1$

- $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq n^k$
- $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} < \left(\frac{ne}{k}\right)^k$
- $\frac{2^{2m}}{2\sqrt{m}} \leq \binom{2m}{m} \leq \frac{2^{2m}}{\sqrt{2m}}$
- $\left(\frac{n}{e}\right)^n \leq n! \leq en\left(\frac{n}{e}\right)^n$

- $(1-p)^m \leq e^{-mp}$ for $p > 0$ small
- $e^{-2p} \leq 1-p$ for $0 \leq p \leq \frac{1}{2}$
- $1+x \leq e^x \quad \forall x \in \mathbb{R}$
- $\left(1+\frac{x}{n}\right)^n < e^x$ for $n \in \mathbb{N}$

Alteration Method

Step1: Random construction
Step2: Get rid of the "bad" parts

Theorem For any n, k, $R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$

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Proof Step 1 Randomly 2-color edges of K_n

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$\mathbb{P}[\text{a fixed } K_k \text{ is monochromatic}] = 2^{1-\binom{k}{2}}$

Let $X = \#$ monochromatic K_k. $\mathbb{E}[X] = \binom{n}{k} 2^{1-\binom{k}{2}}$

By Pigeonhole prop., $\exists$ 2-coloring with atmost these many monochromatic K_k.

By deleting one vertex from each such monochromatic K_k we have 2-coloring on K_t with no mono. K_k & $t = n - \binom{n}{k} 2^{1-\binom{k}{2}}$ ∎

Explicit bound on $R(k,k)$? $R(k,k) > \left(\frac{1}{e} - o(1)\right) k\, 2^{k/2}$

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Since $\binom{n}{k} < \left(\frac{ne}{k}\right)^k$ we have $\underbrace{R(k,k) > n - \left(\frac{ne}{k}\right)^k 2^{1-k(k-1)/2}}_{\text{choose } n \text{ to maximize this}}$

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Differentiating & setting $=0$ gives $1 = k \frac{e}{k} \left(\frac{ne}{k}\right)^{k-1} 2^{1-k(k-1)/2}$

i.e., $n = \frac{1}{e} k 2^{k/2} \underbrace{(2e)^{-1/k-1}}_{\to 1 \text{ when } k \to \infty}$

Choose $n = \left\lceil \frac{1}{e} k 2^{k/2} \right\rceil$ & plug into

$n - \binom{n}{k} 2^{1-\binom{k}{2}}$ & simplify.

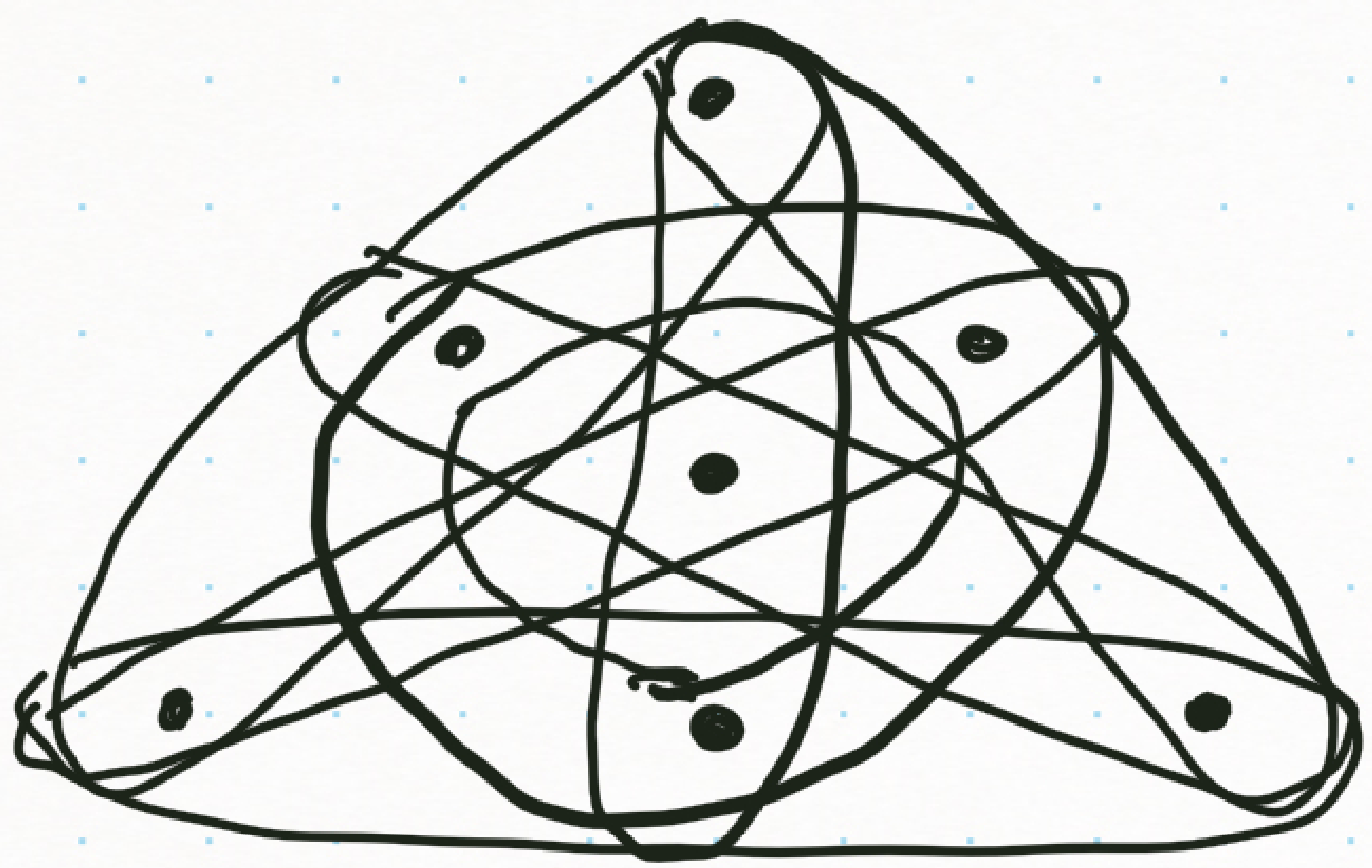
to get $\frac{1}{e} k 2^{k/2} \left(1 - \frac{2e}{k}\right)$

i.e., $\left(\frac{1}{e} - \frac{2}{k}\right) k 2^{k/2}$

Defn A <u>k-uniform hypergraph</u> (k-graph) is $H=(V,E)$ where V (vertices) is a finite set and E (edges) $\subseteq \binom{V}{k}$

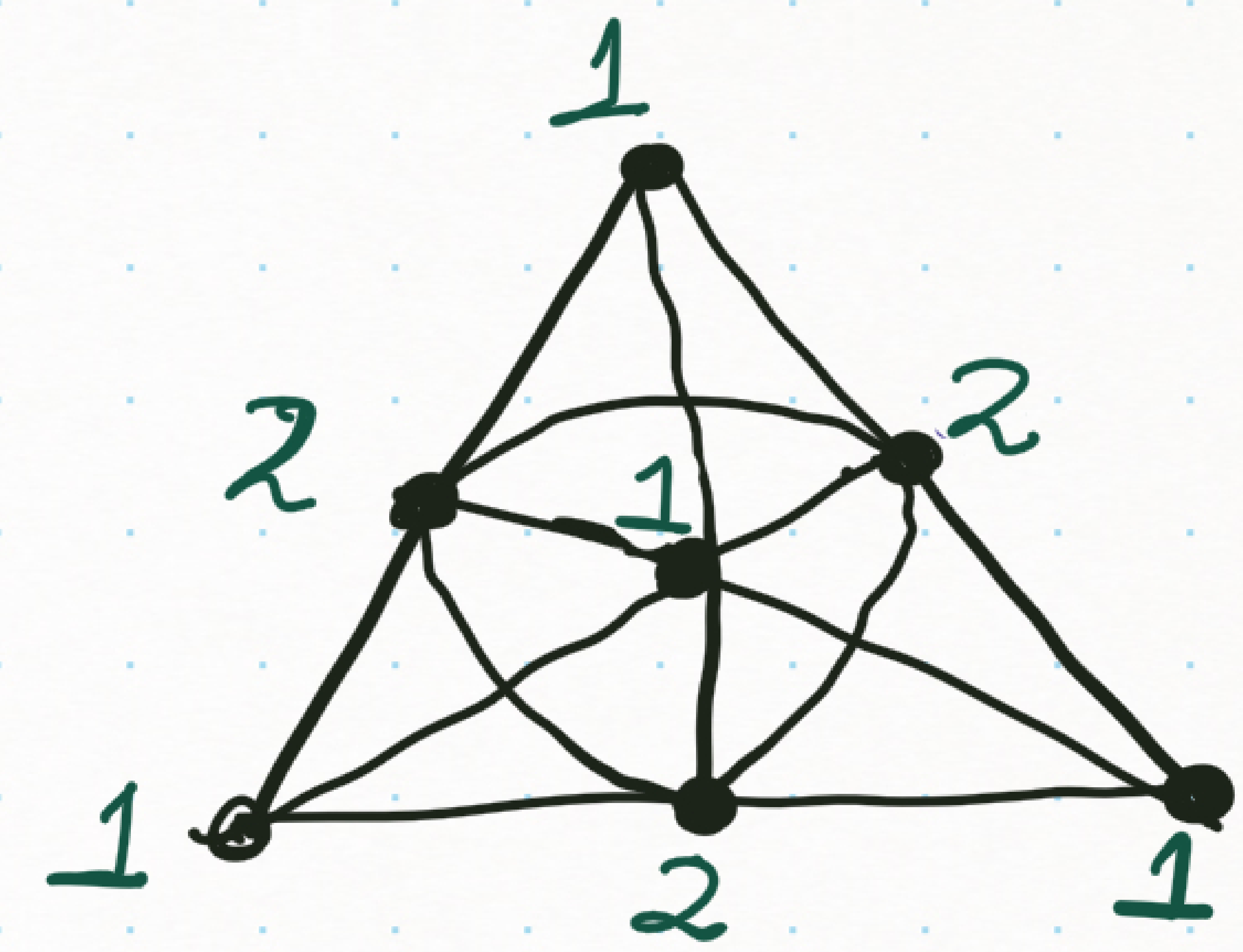
family of k-element subsets of V

Defn H is <u>r-colorable</u> if its vertices can be colored using r colors s.t. no edge is monochromatic.



<u>Fano plane</u>

3-uniform on 7 vertices w. 7 edges



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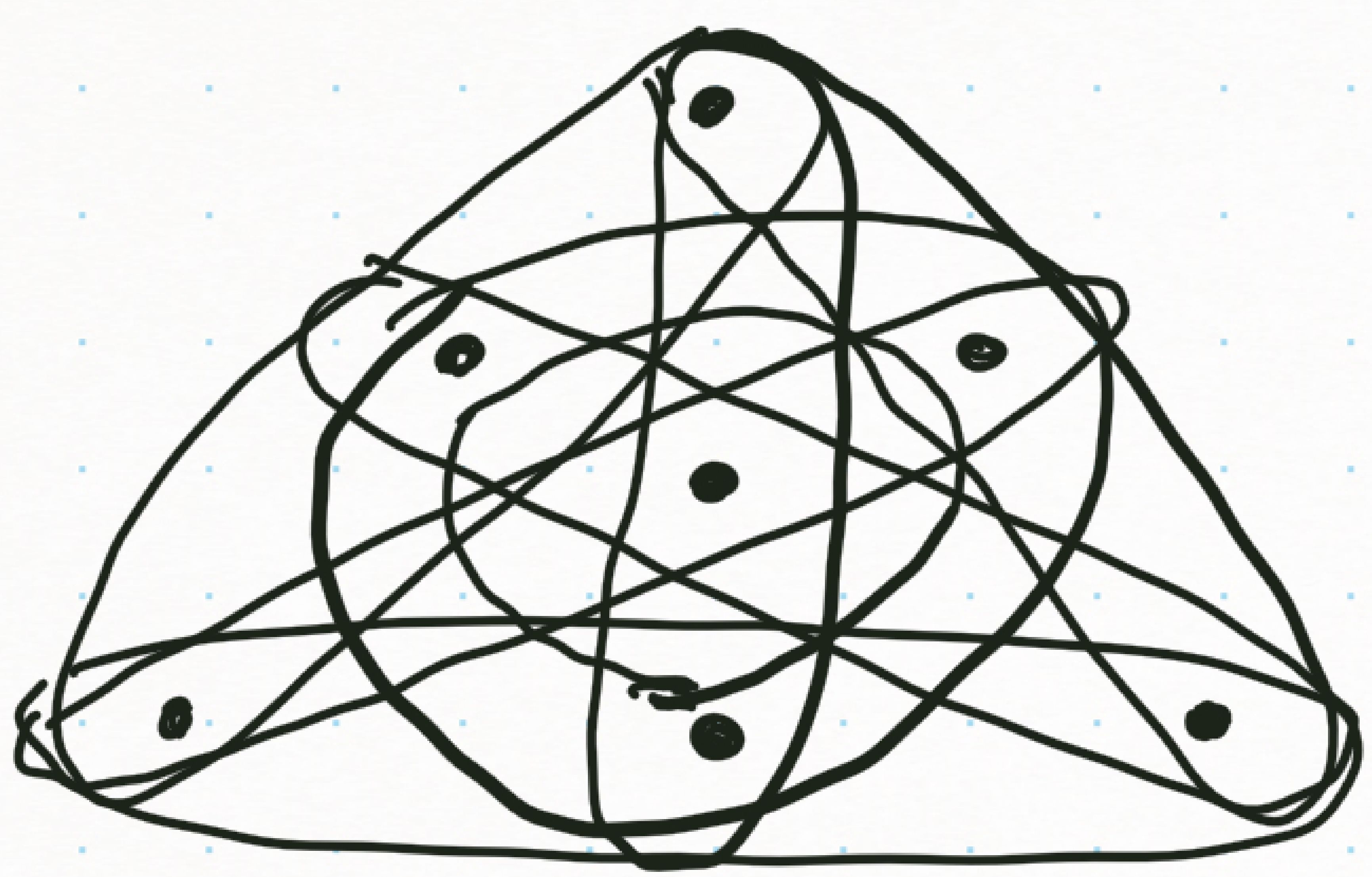

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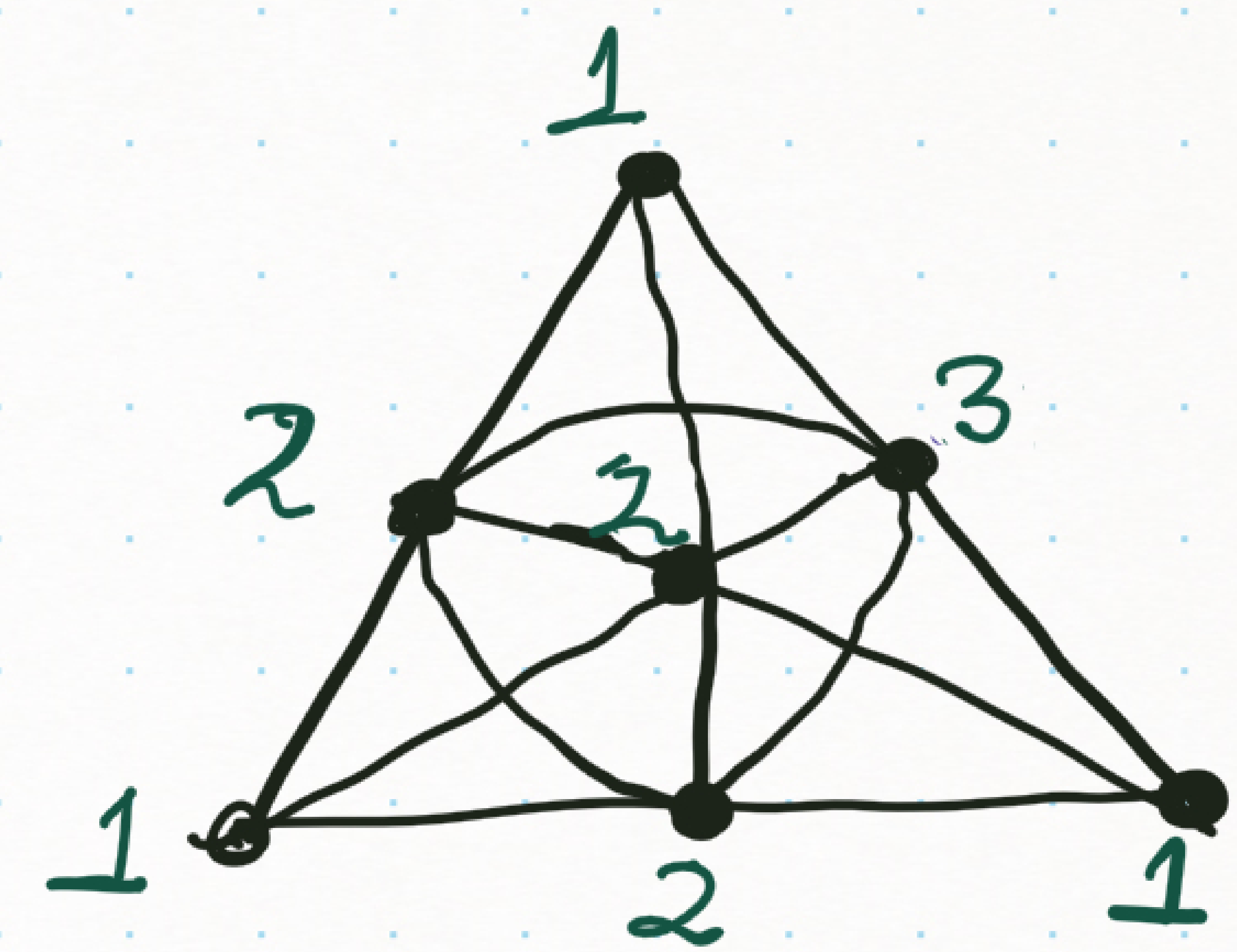

3-colorable but not 2-colorable

Let $m(k)$ = minimum number of edges in a k-uniform hypergraph that is not 2-colorable

"<u>Property B</u>" for Bernstein (1908)

- $m(2) =$

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"Property B" for Bernstein (1908)

- $m(2) = 3$
- $m(3) = 7$
- $m(4) = 23$ (Ostergard 2014)
- $m(k) = ?$ for $k \geq 5$.

Theorem (Erdős 1964) $m(k) \geq 2^{k-1} \quad \forall k \geq 2$

Proof To Show:

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<u>Proof</u> <u>To Show:</u> Every k-uniform hypergraph with fewer than 2^{k-1} edges is 2-colorable.

Let $m < 2^{k-1}$ edges

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Let $m < 2^{k-1}$ edges

In a random 2-coloring, $\mathbb{P}[\exists \text{ monoch. edge}] \leq m \frac{2}{2^k} < 1$ ∎

(Best known lower bd.: $m(k) \geq \Omega(2^k \sqrt{k/\log k})$)

<u>Theorem (Erdős 1964)</u> $m(k) \leq O(k^2 2^k)$

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Given a coloring $f : V \to [2]$, let A_f denote the event that f is a proper coloring (no monochromatic edges)

If f colors a vertices with color 1 & b vertices with color 2 then $\mathbb{P}[S_i \text{ is monochromatic}] =$

note S_i is random while f is fixed

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$$\mathbb{P}[S_i \text{ is monochromatic}] = \frac{\binom{a}{k} + \binom{b}{k}}{\binom{n}{k}} \geq \frac{2\binom{n/2}{k}}{\binom{n}{k}}$$

assume n even for simplicity

Why?

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Why? $\binom{n/2}{k} \leq \dfrac{\binom{a}{k} + \binom{b}{k}}{2}$ where $a+b=n$

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Why? $\binom{n/2}{k} \leq \frac{\binom{a}{k}+\binom{b}{k}}{2}$ where $a+b=n$

$g(x) = \binom{x}{k}$ is a convex function, so

- $\binom{x}{k}+\binom{n-x}{k}$ is minimized when $x=n/2$
- [Jensen's Inequality]

$$g\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) \leq \frac{\sum_{i=1}^{n} g(x_i)}{n}$$

for any convex function g

$$\mathbb{P}[S_i \text{ is monochromatic}] = \frac{\binom{a}{k}+\binom{b}{k}}{\binom{n}{k}} \geq \frac{2\binom{n/2}{k}}{\binom{n}{k}} =$$

$$= \frac{2\,(n/2)(n/2-1)\cdots(n/2-k+1)}{n(n-1)\cdots(n-k+1)}$$

$$\geq 2\left(\frac{n/2-k+1}{n-k+1}\right)^k = 2^{-k+1}\left(1-\frac{k-1}{n-k+1}\right)^k \geq c\,2^{-k}$$

$c>0$ constant

if we set $\boxed{n=2k^2}$

i.e., $2^{-k}\left[2\left(1-\frac{k-1}{2k^2-k+1}\right)^k\right]$

$\therefore\ \mathbb{P}[f \text{ is a proper 2-coloring}]$

$\leq$

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(using

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$$\leq (1-c2^{-k})^m \leq e^{-c2^{-k}m} \quad (\text{using } 1+x \leq e^x \ \forall x)$$

$\therefore \mathbb{P}[\text{There is a proper 2-coloring of } H]$

$$\leq \sum_f \mathbb{P}[f \text{ is a proper 2-coloring}] \leq 2^n e^{-c2^{-k}m}$$

which is < 1 for $m \leq O(k^2 2^k)$

i.e., $\exists\, S_1, S_2, \ldots, S_m$ such that no 2-coloring is proper

The 2-colorability of k-uniform hypergraphs is closely related to k-choosability of bipartite graphs.

Recall In list coloring, each vertex v of G is assigned a list of colors $L(v)$.

G is k-choosable (k-list colorable) if it has a proper coloring f s.t. $f(v) \in L(v)$ $\forall v$ for every list assignment L s.t. $|L(v)| \geq k$.

List chromatic number, $\chi_\ell(G)$ = smallest k such that G is k-choosable.

- $\chi(G) \leq \chi_\ell(G)$ ← Why?

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- $\chi(G) \leq \chi_\ell(G)$
- Show $2 = \chi(K_{3,3}) < \chi_\ell(K_{3,3})$
- $\chi_\ell(G) \leq t$?
- $\chi_\ell(G) > t$?

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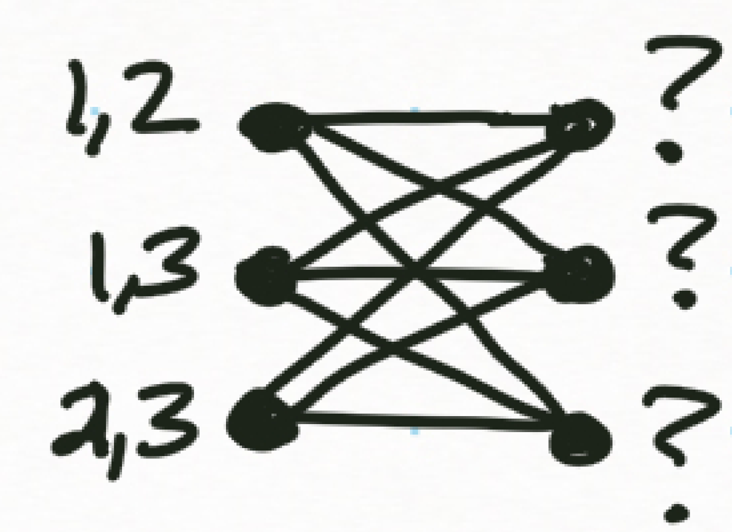

Try

- $\chi_\ell(G) \leq t$? $\forall$ t-list-assignment L, G is L-colorable
- $\chi_\ell(G) > t$? $\exists$ t-list-assignment L s.t. G is not L-colorable

The 2-colorability of k-uniform hypergraphs is closely related to k-choosability of bipartite graphs.

Recall In list coloring, each vertex v of G is assigned a list of colors $L(v)$.
G is k-choosable (k-list colorable) if it has a proper coloring f s.t. $f(v) \in L(v)$ $\forall v$ for every list assignment L s.t. $|L(v)| \geq k$.
List chromatic number, $\chi_\ell(G)$ = smallest k such that G is k-choosable.

- $\chi(G) \leq \chi_\ell(G)$
- Show $2 = \chi(K_{3,3}) < \chi_\ell(K_{3,3})$
- Try $\chi_\ell(K_{k,t}) = k+1 \iff t \geq k^k$

The 2-colorability of k-uniform hypergraphs is closely related to k-choosability of bipartite graphs.

Erdős-Rubin-Taylor 1979 $\quad m(k) \leq n(k) \leq 2m(k)$

where $m(k) = \min$ # edges in a non-2-colorable k-unif. hyperg.

$n(k) = \min$ # vertices in a non-k-choosable bipartite graph

This gives $\rightarrow$

Cor $\quad \chi_\ell(K_{n,n}) = (1+o(1))\log_2 n$

Do HW#1

Defn A $\underbrace{\text{family } \mathcal{F} \text{ of sets}}$ is <u>intersecting</u> if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$

"Edges of a hypergraph"

Ques What is the largest size of intersecting family of subsets of $[n]$?

"Edges of a hypergraph"

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<u>Ques</u> What is the largest size of intersecting family of subsets of $[n]$?

2^{n-1} ("Don't pick complementary pairs of sets")

<u>More interesting question</u> What is the largest intersecting family of k-subsets of $[n]$? $\boxed{\max |\mathcal{F}| \text{ for } \mathcal{F} \subseteq \binom{[n]}{k}}$

<u>Theorem [Erdős–Ko–Rado 1961 (proved in 1938)]</u>

If $\underline{n \geq 2k}$, $\mathcal{F} \subseteq \binom{[n]}{k}$, $\mathcal{F}$ is intersecting then $|\mathcal{F}| \leq \binom{n-1}{k-1}$

Why? What if $n \leq 2k-1$?

Sharpness?

Defn A $\overbrace{\text{family } \mathcal{F} \text{ of sets}}^{\text{"Edges of a hypergraph"}}$ is *intersecting* if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$

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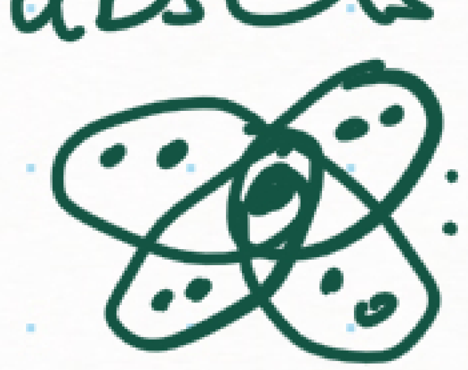
Sharpness? "Sunflower" All k-subsets of $[n]$ containing 1



Erdős–Ko–Rado

If $n \geq 2k$, $\mathcal{F} \subseteq \binom{[n]}{k}$, $\mathcal{F}$ intersecting then $|\mathcal{F}| \leq \binom{n-1}{k-1}$

Proof

Consider a uniformly random circular permutation of $1, 2, \ldots, n$ (arrange $[n]$ randomly in a circle)

For each $A \in \binom{[n]}{k}$, we say A is contiguous if all elements of A lie in a contiguous block on the circle.

$\mathbb{P}[A \text{ is contiguous}] = n / \binom{n}{k}$ $\leftarrow$ why?

$\therefore$ Expected number of contiguous sets in $\mathcal{F}$ is exactly $\dfrac{n|\mathcal{F}|}{\binom{n}{k}}$

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$\therefore$ Expected number of contiguous sets in $\mathcal{F}$ is exactly $\dfrac{n|\mathcal{F}|}{\binom{n}{k}}$

Since $\mathcal{F}$ is intersecting, there are atmost k contiguous sets in $\mathcal{F}$ (under every circular ordering of $\mathcal{F}$). why?

Suppose $A \in \mathcal{F}$ is contiguous. Then there are $2(k-1)$ other contiguous sets (not neccessarily in $\mathcal{F}$) that intersect A, but they can be paired off into disjoint pairs since $n \geq 2k$.

Since $\mathcal{F}$ is intersecting, we can only include $k-1$ more such sets.

Erdős–Ko–Rado

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Since $\mathcal{F}$ is intersecting, there are atmost k contiguous sets in $\mathcal{F}$ (under every circular ordering of $\mathcal{F}$).

Hence $\exists$ circular ordering of $\mathcal{F}$ with $\dfrac{n|\mathcal{F}|}{\binom{n}{k}}$ contiguous sets

$\therefore \dfrac{n|\mathcal{F}|}{\binom{n}{k}} \leq k$ i.e., $|\mathcal{F}| \leq \dfrac{k}{n}\binom{n}{k} = \binom{n-1}{k-1}$ $\blacksquare$

Erdös-Ko-Rado If $n \geq 2k$, $\mathcal{F} \subseteq \binom{[n]}{k}$, $\mathcal{F}$ intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$

Proof (Alternate)

Lemma Let $X = \{0, 1, \ldots, n-1\}$ with addition modulo n. Define $A_s = \{s, s+1, \ldots, s+k-1\} \subseteq X$ for $0 \leq s \leq n-1$. Then for $n \geq 2k$, any intersecting family $\mathcal{F} \subseteq \binom{X}{k}$ contains at most k of the sets A_s.

Proof If $A_i \in \mathcal{F}$ then any other $A_s \in \mathcal{F}$ must be one of $A_{i-k+1}, A_{i-k}, \ldots\ldots, A_{i-1}$ or $A_{i+1}, \ldots\ldots, A_{i+k-1}$.
These are $2k-2$ sets that can be divided into $k-1$ pairs of set: (A_s, A_{s+k}). Since $n \geq 2k$, $A_s \cap A_{s+k} = \phi$ and we can pick atmost one of each such pair.

Erdös-Ko-Rado If $n \geq 2k$, $\mathcal{F} \subseteq \binom{[n]}{k}$, $\mathcal{F}$ intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$

Proof

Lemma Let $X = \{0, 1, \dots, n-1\}$ with addition modulo n. Define $A_s = \{s, s+1, \dots, s+k-1\} \subseteq X$ for $0 \leq s \leq n-1$. Then for $n \geq 2k$, any intersecting family $\mathcal{F} \subseteq \binom{X}{k}$ contains at most k of the sets A_s.

We may assume $\mathcal{F}$ is intersecting family in $\binom{X}{k}$ with $X = \{0, \dots, n-1\}$

For permutation $\sigma : X \to X$ (bijection), define $\sigma(A_s) = \{\sigma(s), \sigma(s+1), \dots, \sigma(s+k-1)\}$ w. addition mod n.

By Lemma, at most k of these n sets $\sigma(A_s)$ are in $\mathcal{F}$.

So, if we choose s randomly & σ independently and uniformly,

$\mathbb{P}[\sigma(A_s) \in \mathcal{F}] \leq k/n$ $\leftarrow$ Here underlying probability space is $\{0, 1, \dots, n-1\} \times S_n$ with uniform measure

The choice of $\sigma(A_s)$ is equivalent to random choice of k-subset of X

i.e, $\mathbb{P}[\sigma(A_s) \in \mathcal{F}] = |\mathcal{F}| / \binom{n}{k}$. $\quad \therefore |\mathcal{F}| / \binom{n}{k} \leq k/n$ i.e. $|\mathcal{F}| \leq \binom{n-1}{k-1}$ ∎