

Math 554

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Setup for Lovász Local Lemma:

Defn An event B is mutually independent of events A_1, \dots, A_k

$$\text{if } \mathbb{P}[B | C_{i_1} \wedge C_{i_2} \wedge \dots \wedge C_{i_\ell}] = \mathbb{P}[B] \quad \forall \ell \quad \forall 1 \leq i_1 < i_2 < \dots < i_\ell \leq k.$$

where each C_{i_j} is either A_{i_j} or $\overline{A_{i_j}}$.

Defn G is said to be a dependency graph of events A_1, \dots, A_n

if $V(G) = [n]$ and for each i , A_i is mutually independent of all events A_j s.t. $ij \notin E(G)$.

Caution! mutual independence \neq pairwise independence.

e.g. Pick $x_1, x_2, x_3 \in \{0, 1\}$ uniformly & independently at random.

For $i=1, 2, 3$, let A_i be event $x_{i+1} + x_{i+2} = 0 \pmod{2}$ (indices $\pmod{3}$)

Then these events are pairwise independent but not mutually ind.

$\overline{K_3}$ is not a valid dep. graph, but is a valid dep. graph.



However, in many (most?) applications of LLL, the underlying probability space is a product probability space i.e., it is based on a collection of independent random experiments and each event A_i is determined by a subcollection S_i of these independent experiments.

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Mutual Independence Principle Let Z_1, Z_2, \dots, Z_m be independent experiments and A_1, \dots, A_n be events s.t. the occurrence of each A_i is determined by a subset S_i of Z_1, \dots, Z_m .
If S_i is disjoint from S_{j_1}, \dots, S_{j_k} then A_i is mutually ind. of $\{A_{j_1}, \dots, A_{j_k}\}$

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In this set-up, a valid dependency graph can be formed by placing an edge $i \leftrightarrow j$ based on $|S_i \cap S_j| > 0$.

e.g. in our application of LLL to $R(K, K)$, Z_1, \dots, Z_m are the independent "coin flips" used to color m edges of K_n . A_S is independent of all $A_{S'}$ when S & S' share no edge, $|S \cap S'| = 0$.

Lovász Local Lemma, General Form

Let A_1, \dots, A_n be events. For each $i \in [n]$, let $N(i) \subseteq [n]$ be such that A_i is independent of $\{A_j : j \notin \{i\} \cup N(i)\}$.

neighborhood in
a dependency graph

if $x_1, \dots, x_n \in [0, 1)$ satisfy

$$P[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j) \quad \forall i \in [n]$$

then with probability $\geq \prod_{i=1}^n (1 - x_i)$, none of the events A_i occur.

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LLL, Symmetric Form

Let G be a dependency graph of events A_1, \dots, A_n with $\Delta(u) \leq d$ and $P[A_i] \leq p \quad \forall i$. Then,

$$e p (d+1) < 1 \Rightarrow P\left[\bigwedge_{i=1}^n \bar{A}_i\right] > 0$$

Proof Set $x_i = 1/(d+1) < 1 \quad \forall i$

Then $x_i \prod_{j \in N(i)} (1 - x_j) \geq \frac{1}{d+1} \left(1 - \frac{1}{d+1}\right)^d > \frac{1}{(d+1)e} \geq p$

← neighborhood in dep. graph.

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$$e p (d+1) < 1 \Rightarrow P\left[\bigwedge_{i=1}^n \bar{A}_i\right] > 0$$

e is sharp; cannot be replaced by any smaller constant
often, checking $4p(d+1) < 1$ is enough.

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Proof

Claim $P[A_i | \bigwedge_{j \in S} \bar{A}_j] \leq x_i$ whenever $i \notin S \subseteq [n]$

With this claim, we can easily finish the proof:

$$\begin{aligned} P[\bar{A}_1 \wedge \bar{A}_2 \wedge \dots \wedge \bar{A}_n] &= P[\bar{A}_1] P[\bar{A}_2 | \bar{A}_1] P[\bar{A}_3 | \bar{A}_1, \bar{A}_2] \dots P[\bar{A}_n | \bar{A}_1, \dots, \bar{A}_{n-1}] \\ &\geq ? \end{aligned}$$

neighborhood in a dependency graph



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neighborhood in a dependency graph



Claim $P[A_i | \bigwedge_{j \in S} \bar{A}_j] \leq x_i$ whenever $i \notin S \subseteq [n]$

Proof by induction on $|S|$. $|S|=0$ ✓

Let $i \notin S$. Denote $S_1 = S \cap N(i)$ and $S_2 = S \setminus S_1$, so

$$P[A_i | \bigwedge_{j \in S} \bar{A}_j] = P[A_i \bigwedge_{j \in S_1} \bar{A}_j | \bigwedge_{j \in S_2} \bar{A}_j] / P[\bigwedge_{j \in S_1} \bar{A}_j | \bigwedge_{j \in S_2} \bar{A}_j] =: X/Y$$

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$$X \leq P[A_i | \bigwedge_{j \in S_2} \bar{A}_j] = P[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j)$$

Set $S_1 = \{j_1, \dots, j_r\}$, so

$$Y = P[\bar{A}_{j_1} | \bigwedge_{j \in S_2} \bar{A}_j] P[\bar{A}_{j_2} | \bar{A}_{j_1} \bigwedge_{j \in S_2} \bar{A}_j] \dots P[\bar{A}_{j_r} | \bar{A}_{j_1} \dots \bar{A}_{j_{r-1}} \bigwedge_{j \in S_2} \bar{A}_j]$$

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$$\geq (1 - x_{j_1}) (1 - x_{j_2}) \dots (1 - x_{j_r}) \quad \text{Why?}$$

$$\geq \prod_{j \in N(i)} (1 - x_j) \quad \because ?$$

Claim $P[A_i | \bigwedge_{j \in S} \bar{A}_j] \leq x_i$ whenever $i \notin S \subseteq [n]$

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$$\geq (1 - x_{j_1}) (1 - x_{j_2}) \dots (1 - x_{j_r}) \quad \text{Why? Induction hypothesis}$$

$$\geq \prod_{j \in N(i)} (1 - x_j) \quad \because S_1 \subseteq N(i).$$

A few comments on LLL

① Lopsided Lovasz Local Lemma

Note the following step in proof of LLL

$$X \leq \mathbb{P}[A_i | \bigwedge_{j \in S_2} \bar{A}_j] = \mathbb{P}[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j)$$

The proof would work even if changed to " \leq ".
i.e, we replace the independence assumption on A_i vs. A_j ^{non-neighb} of A_i
with "each A_i is non-negatively dependent on its non-neighbors"

That is, we have a negative dependency graph where
intuitively (!) neighbors of each event consist of events
that are negatively dependent on it.

Modify LLL to LLLL by replacing the independence condition by
 $\mathbb{P}[A_i | \bigwedge_{j \in S} \bar{A}_j] \leq \mathbb{P}[A_i]$ $\forall i \in [n]$ and $S \subseteq [n] \setminus (\{i\} \cup N(i))$

Erdős & Spencer 1991 applied LLLL to show:

Theorem Every $n \times n$ array where each entry appears at most $n/4e$ times has a Latin transversal.

where,

- a transversal of an $n \times n$ array is a set of n entries with one in each row and column.

- a Latin transversal is a transversal with

distinct entries

e.g.

1	2	3	A	K	J	Q
2	3	1	Q	J	K	A
3	1	2	J	Q	A	K
			K	A	Q	J

(Euler's)
 $n \times n$ array with
each symbol appearing
exactly once in each row & column.

Conjecture (Ryser 1967)

Each Latin square of odd order has at least one transversal

Balasubramanian 1990 showed: every L.S. of even order has even # transversals.

A few comments on LLL

② Algorithmic Lovasz Local Lemma

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LLL tells us a "good structure" exists but the guaranteed probability is often very small. How can we sample such a "good structure" efficiently?

Moser (2009) & Moser-Tardos (2010) came up with the first efficient LLL algorithm in the "Product Probability Space" setup.

↳ events are based on underlying independent random experiments

Moser-Tardos LLL algorithm

Initialize all random variables.

While "there are violated events" Do → Pick an arbitrary violated event and resample its variables.

Moser-Tardos 2010 $\exists \{x_i \in [0, 1)\}$ s.t. $P[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j) \forall i \in [n]$

The above randomized algorithm runs in

expected number of steps at most $\sum_i x_i / (1 - x_i)$.

Note that the Moser-Tardos algorithm is a Las Vegas algorithm with polynomial expected runtime [randomized algo that always terminates successfully but runtime might be long]

Other type of randomized algorithm is a Monte Carlo Algo, which always runs in bounded time but may return a bad result with some small probability.

A LV algo can be converted to a MC algo by terminating the algo after some steps (larger than expected #) & applying Markov's ineq. to bound probability of failure.

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Work on algorithmic LLL began with Beck in 1991 with an algorithm for finding a proper 2-coloring of a ^{k-uniform} hypergraph where each edge intersects $< 2^{k/48}$ other hyperedges.
(& since 2010, an explosion: see Entropy Compression Method Local Cut Lemma)

We saw earlier: every k -uniform hypergraph with $< 2^{k-1}$ edges is 2-colorable.

Theorem A k -uniform hypergraph is 2-colorable if every edge intersects less than $\frac{2^{k-1}}{e} - 1$ other edges.

Proof Experiment?

Bad events?

Dependency?

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Theorem A k -uniform hypergraph is 2-colorable if every edge intersects less than $\frac{2^{k-1}}{e} - 1$ other edges.

Proof Experiment? Color each vertex independently & u.a.r. with blue or red.

Bad events? For each edge E , $A_E \equiv$ event E is monochromatic. $\mathbb{P}(A_E) = 2^{1-k} = p$

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Dependency? By Mutual Independence Principle (MIP), each A_f is mutually independent of every $A_{f'}$ where $f \cap f' = \emptyset$

Since each f intersects less than $2^{\frac{k-1}{e}} - 1$ other f' , we can pick $d < 2^{\frac{k-1}{e}} - 1$ i.e., $ep(d+1) < 1$ & LLL applies

and \exists 2-coloring of the hypergraph. 

In applications of LLL, even if the random process is the most obvious one, the choice of "bad events" can be subtle.

We know every graph G contains an ind. set of size $\geq \frac{|V(G)|}{\Delta(G)+1}$

Why? How?

In applications of LLL, even if the random process is the most obvious one, the choice of "bad events" can be subtle.

We know every graph G contains an ind. set of size $\geq \frac{|V(G)|}{\Delta(G)+1}$

By decreasing the desired size of ind. set by a constant factor ($2e$), we can guarantee a structure on the independent set.

Why? How?
 $\sum_{i=1}^k (d_i+1) \geq n$
i.e. $k(\Delta(G)+1) \geq n$

Lemma Let V_1, \dots, V_k be a partition of $V(G)$ with each $|V_i| \geq 2e \Delta(G)$. Then there exists an independent set in G consisting of one vertex from each V_i .

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WLOG, we may assume $|V_i| = \lceil 2e\Delta(G) \rceil = a$, by removing excess vertices from each V_i .

Random Procedure? Pick $v_i \in V_i$ uniformly at random, independently to form a random set S .

Bad Events?

Attempt 1

$A_{i,j} \equiv$ " v_i & v_j have an edge between them"
 for each $1 \leq i < j \leq t$

vertex chosen from V_i vertex chosen from V_j

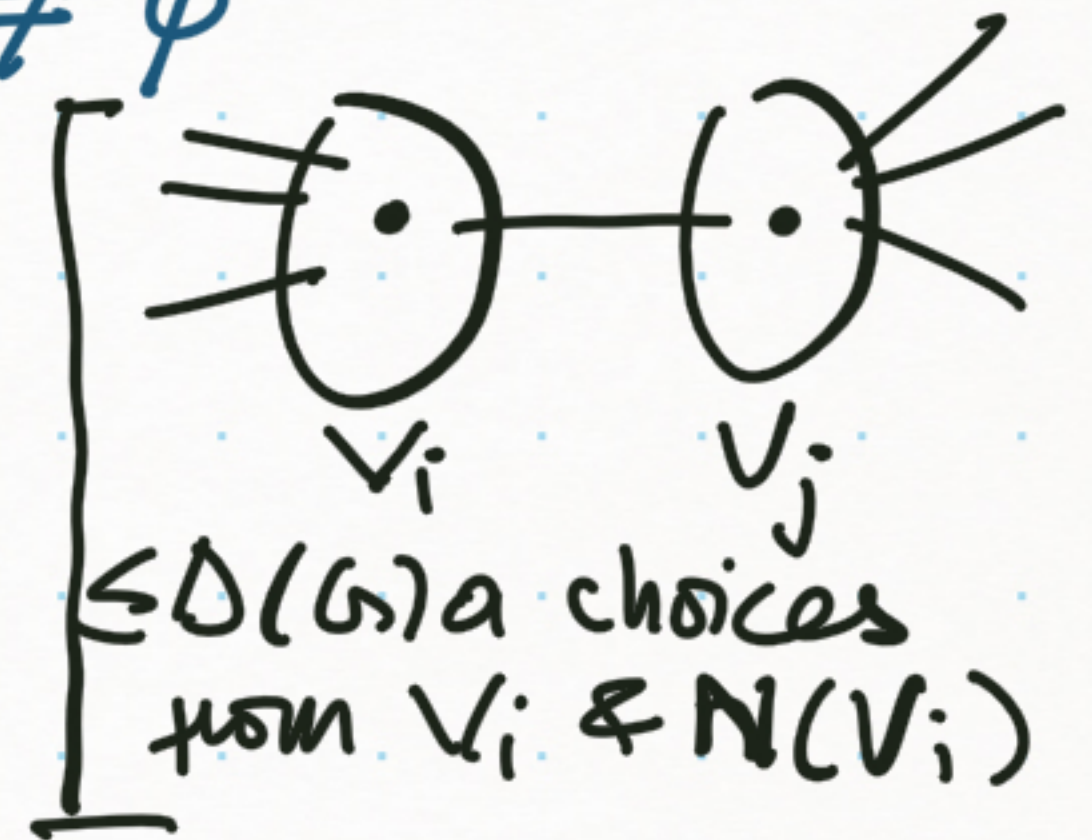
$$P[A_{i,j}] \leq \Delta(G)/a = p$$

Dependency graph: $A_{i,j} \leftrightarrow A_{k,l}$ if $\{i,j\} \cap \{k,l\} \neq \emptyset$

$$\text{max degree of Dep. graph} \leq 2\Delta(G)a = d$$

Then, $e_p(d+1) \approx 2e (\Delta(G))^2 \gg 1$

common endpt. between $v_i v_j$ & $v_k v_l$



Bad Events?

vertex chosen from V_i
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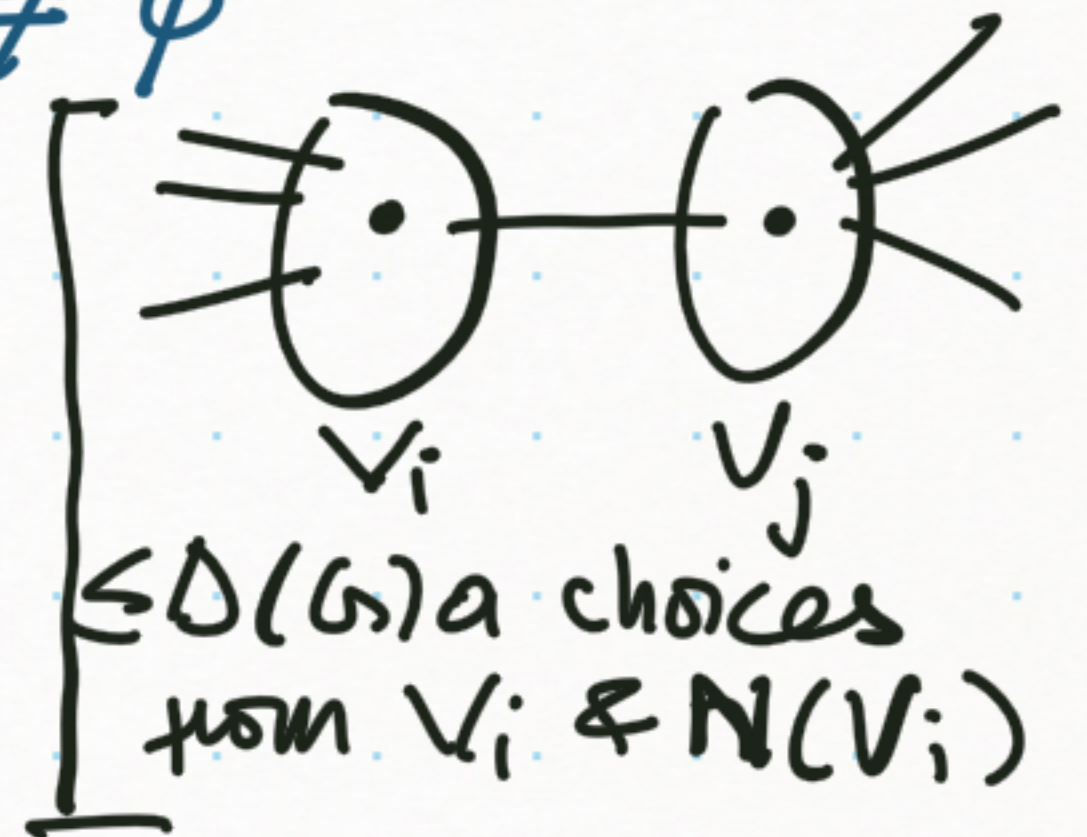
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Attempt 2 $A_e \equiv$ "both endpoints of e chosen"

for each $e \in E(G)$ with endpts. in distinct parts.

$$P[A_e] =$$

common endpt. between $v_i v_j$ & $v_k v_l$



Bad Events?

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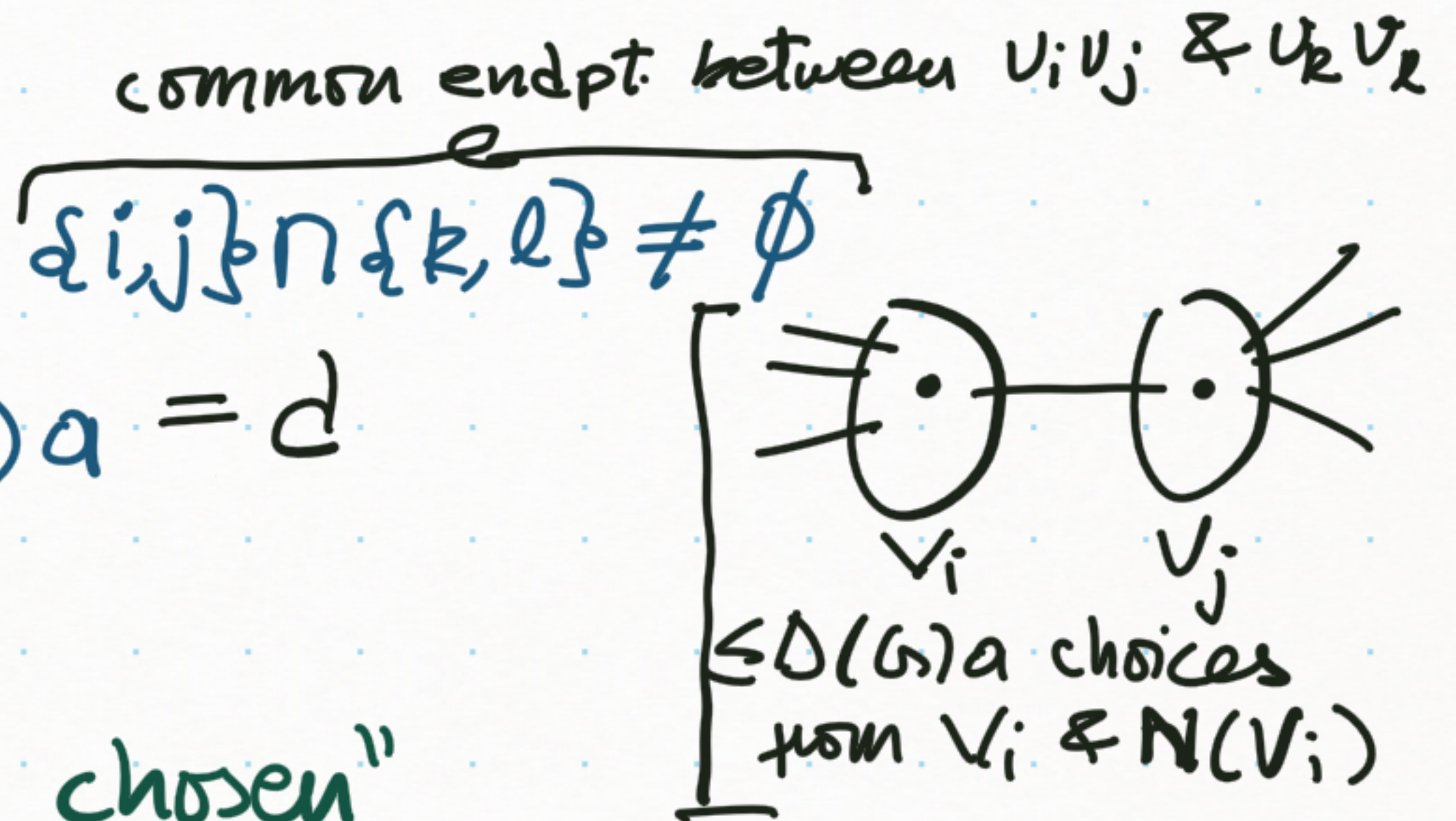
Dependency graph: $A_{i,j} \leftrightarrow A_{k,l}$ if $\{i,j\} \cap \{k,l\} \neq \emptyset$
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Attempt 2 $A_e \equiv$ "both endpoints of e chosen"
 for each $e \in E(G)$ with endpts. in distinct parts.

$$P[A_e] = \frac{1}{a^2} = p$$

Dependency graph: By MIP, A_e is mutually independent of the events corresponding to edges outside $V_i \cup V_j$ where $u \in V_i$ & $v \in V_j$
 $\therefore d \leq 2\Delta(G)a$ and $e_p(d+1) \leq e \frac{1}{a^2} 2\Delta(G)a = \frac{2e\Delta(G)}{a} < 1$.



Thm (Reed 1999) Let L be a list assignment for a graph G
s.t. $|L(v)| \geq k$ for all v . If every color in the palette $\bigcup_{v \in V(G)} L(v)$
appears in at most $k/2e$ of the lists in each vertex
neighborhood, then G has a proper L -coloring.

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Proof WLOG $|L(v)| = k$

For each $v \in V(G)$, choose a color u.a.s. from $L(v)$, independently

Bad Events? $A_{uv} \equiv$ event both u & v receive the same color for all $uv \in E(G)$.

Check LLL fails to apply with this definition.

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For each edge uv and colors $c \in L(u) \cap L(v)$, define

$A_{uv,c}$ \equiv event u & v both receive color c .

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$A_{uv,c}$ \equiv event u & v both receive color c .

$$P[A_{uv,c}] = \frac{1}{k^2} = p.$$

By MIP, $A_{uv,c}$ is mutually ind. of set of all such events whose edges are disjoint from u,v .

To bound d , we need to count events corresponding to edges incident to edge uv .

For $u'v'$ incident to uv , the event $A_{u'v',c'}$ is defined

$$\text{iff } c' \in L(u') \cap L(v')$$

Suppose $u = u'$ & $v \neq v'$,

fixed $u' = u$ & $v' \in N(u)$
 $\underbrace{\quad}_{\{v\}}$



then at most \cdot ?

such events exist
for each $c' \in L(u') = L(u)$

For $u'v'$ incident to uv , the event $A_{u'v',c}$ is defined

$$\text{iff } c' \in L(u') \cap L(v')$$

Suppose $u=u'$ & $v \neq v'$,

fixed $u'=u$ & $v' \in N(u) - \{v\}$



then at most $\frac{k}{2e} - 1$

such events exist
for each $c' \in L(u') = L(u)$
(excluding $c \in L(u) \cap L(v)$
(for the event $A_{uv,c}$))

Similarly, when $v=v'$ & $u \neq u'$

\leftarrow # colors in each list

$$\therefore d \leq k \cdot 2 \cdot \left(\frac{k}{2e} - 1\right)$$

\uparrow 2 possibilities
for $u=u'$ & $v=v'$

$$\text{Then, } e p(d+1) \leq e \frac{1}{k^2} \left(2k \left(\frac{k}{2e} - 1\right) + 1\right) = 1 - \frac{2e}{k} + \frac{e}{k^2}$$

$$= 1 - \left(\frac{2ek - e}{k^2}\right) < 1$$

$\frac{k}{2e}$ was improved to $\frac{k}{2}$ [Maxwell 2001] & $k(1-o(1))$ [Reed-Sudakov 2002]
Reed conjectured that $(k-1)$ suffices. ■

Lovász Local Lemma, General Form

Let A_1, \dots, A_n be events. For each $i \in [n]$, let $N(i) \subseteq [n]$ be such that A_i is independent of $\{A_j : j \notin \{i\} \cup N(i)\}$.

If $x_1, \dots, x_n \in [0, 1)$ satisfy

$$P[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j) \quad \forall i \in [n]$$

then with probability $\geq \prod_{i=1}^n (1 - x_i)$, none of the events A_i occur.

LLL, Symmetric Form

Let G be a dependency graph of events A_1, \dots, A_n with $\Delta(u) \leq d$ and $P[A_i] \leq p \quad \forall i$. Then,

$$e p (d+1) < 1 \Rightarrow P\left[\bigwedge_{i=1}^n \bar{A}_i\right] > 0$$

Symmetric LLL can be useless when there are different kinds of "bad" events with varying probabilities.

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LLL, Asymmetric Form (aka Neighborhood Local Lemma)

Let A_1, \dots, A_n be events. Let $N(i)$ be the neighborhood of A_i in a dependency graph, as defined above.

if $P[A_i] < \frac{1}{2}$ and $\sum_{j \in N(i)} P[A_j] \leq \frac{1}{4} \quad \forall i$ then $P[\bigwedge_{i=1}^n \bar{A}_i] > 0$.

Proof set $x_i = 2P[A_i] \quad \forall i$ in LLLG. (so $x_i \in [0, 1)$)

Then, $x_i \prod_{j \in N(i)} (1 - x_j) \geq x_i (1 - \sum_{j \in N(i)} x_j) = 2P[A_i] (1 - \sum_{j \in N(i)} 2P[A_j]) \geq P[A_i]$

↑ why? *↑ why?*

Lovász Local Lemma, General Form

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More than one type of "bad" event with different probabilities,
But few high-probability events in each neighborhood $N(i)$.

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Apply LLL (Symmetric) & LLL (Asymmetric) to the following:

Let H be a hypergraph with $|e| \geq 3 \quad \forall e \in E(H)$.

If $\sum_{\substack{f \in E(H) \\ \text{s.t. } e \cap f \neq \emptyset}} 2^{-|f|} \leq \frac{1}{8} \quad \forall e \in E(H)$, then H is 2-colorable.

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2-color vertices uniformly at random, independently.

$A_e \equiv$ event e is monochromatic

$$P[A_e] = 2^{-|e|+1} \leq 2^{-3+1} = \frac{1}{4}; \quad \sum_{\substack{f \in E(H) \\ e \cap f \neq \emptyset}} P[A_f] = \sum_{\substack{f \in E(H) \\ e \cap f \neq \emptyset}} 2^{-|f|+1} = 2 \sum_{\substack{f \in E(H) \\ e \cap f \neq \emptyset}} 2^{-|f|} \leq \frac{1}{4} \quad \forall e$$

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Defn A proper coloring of a graph \mathcal{G} is called β -frugal if no color appears more than β times in the neighborhood of any vertex.

What is 1-frugal coloring of \mathcal{G} ?

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Defn A proper coloring of a graph G is called β -frugal if no color appears more than β times in the neighborhood of any vertex.

What is 1-frugal coloring of G ?

No color is repeated in $N(v) \forall v$, i.e. proper coloring if each neighborhood is converted into a clique.

That is, G has 1-frugal coloring $\iff G^2$ has a proper coloring using k colors

How many colors does a β -frugal coloring need?

Theorem [Hind, Mollay, Reed 1997]

If G has max degree $\Delta \geq \beta^{\beta}$, then G has a β -frugal coloring with at most $16 \Delta^{1+1/\beta}$ colors.

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Proof $\beta=1$ G has 1-frugal coloring if G^2 has a proper coloring.

By Brooks, G^2 can be properly colored with $\Delta(G^2)+1$ colors
i.e., at most $\Delta(G)^2+1$ colors

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Note $16 \Delta^{1+1/\beta} = 16 \Delta^2 \geq \Delta^2+1$.

Assume $\beta \geq 2$ Assign a color to each vertex u.a.r from
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We have two types of bad events \longrightarrow not proper coloring
 \longrightarrow not β -frugal

& Each type has different probabilities and different dependencies.

Type-1 events $A_{uv} \equiv u, v$ have same color, for each $uv \in E(G)$

Type-2 events $B_{u_1, \dots, u_{\beta+1}} \equiv u_1, \dots, u_{\beta+1}$ have same color,
for each set of $\beta+1$ neighbors, $u_1, \dots, u_{\beta+1}$, of some vertex.

For any k , probability of k vertices having same color is $1/Q^{k-1}$, so

$$P[A_{uv}] = 1/Q \quad \text{and} \quad P[B_{u_1, \dots, u_{\beta+1}}] = 1/Q^\beta$$

What about dependencies?

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Each Type-1 event depends on at most 2Δ Type-1 events
and Type-2 events.



by MIP, at most $(\Delta-1) + (\Delta-1)$
such events

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Each Type-1 event depends on at most 2Δ Type-1 events and $2\Delta \binom{\Delta-1}{\beta}$ Type-2 events.



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By MIP, $B_{u_1, \dots, u_{\beta+1}}$ depends on A_{uv} only if $\{u_1, \dots, u_{\beta+1}\} \cap \{u, v\} \neq \emptyset$



2 choices (u or v) for common vertex ($u_i = u$ or v)
 Δ choices among neighbors of common vertex (say u) for x
 $\binom{\Delta-1}{\beta}$ choices among remaining neighbors of x for the remaining $\{u_1, \dots, u_{\beta+1}\}$

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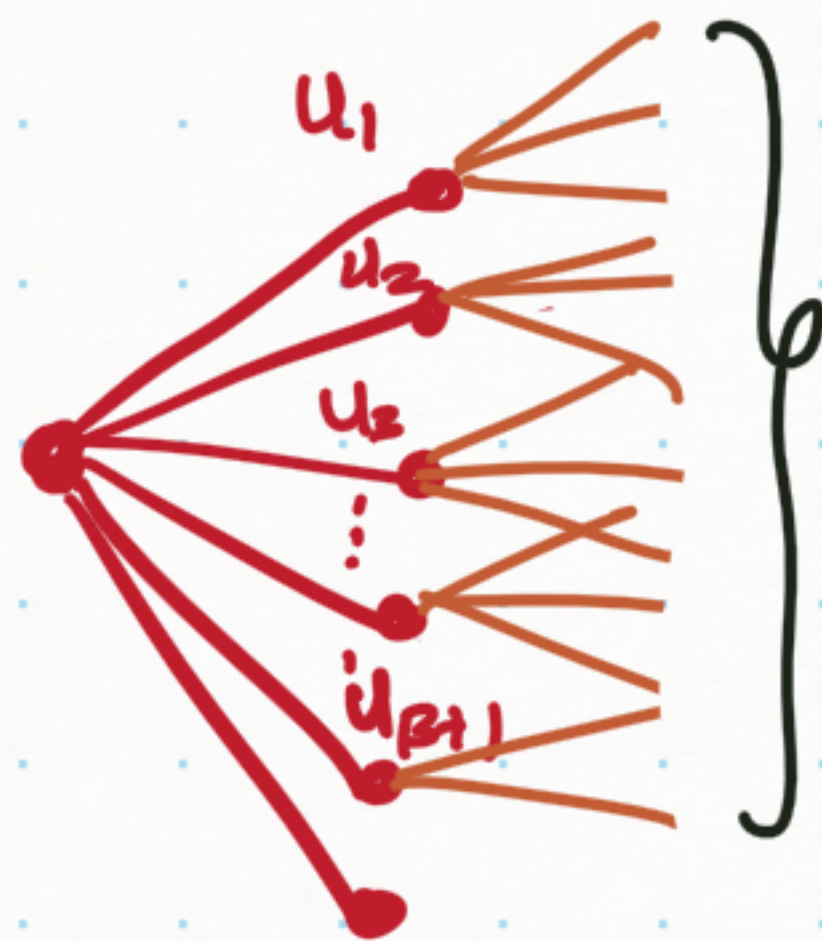
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Each Type-2 event depends on at most $(\beta+1)\Delta$ Type-1 events
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at most
 $(\beta+1)(\Delta-1)$ edges (to use as uv_i in type-1 event)
incident to $u_1, \dots, u_{\beta+1}$

Type-1 events $A_{uv} \equiv u, v$ have same color, for each $uv \in E(G)$

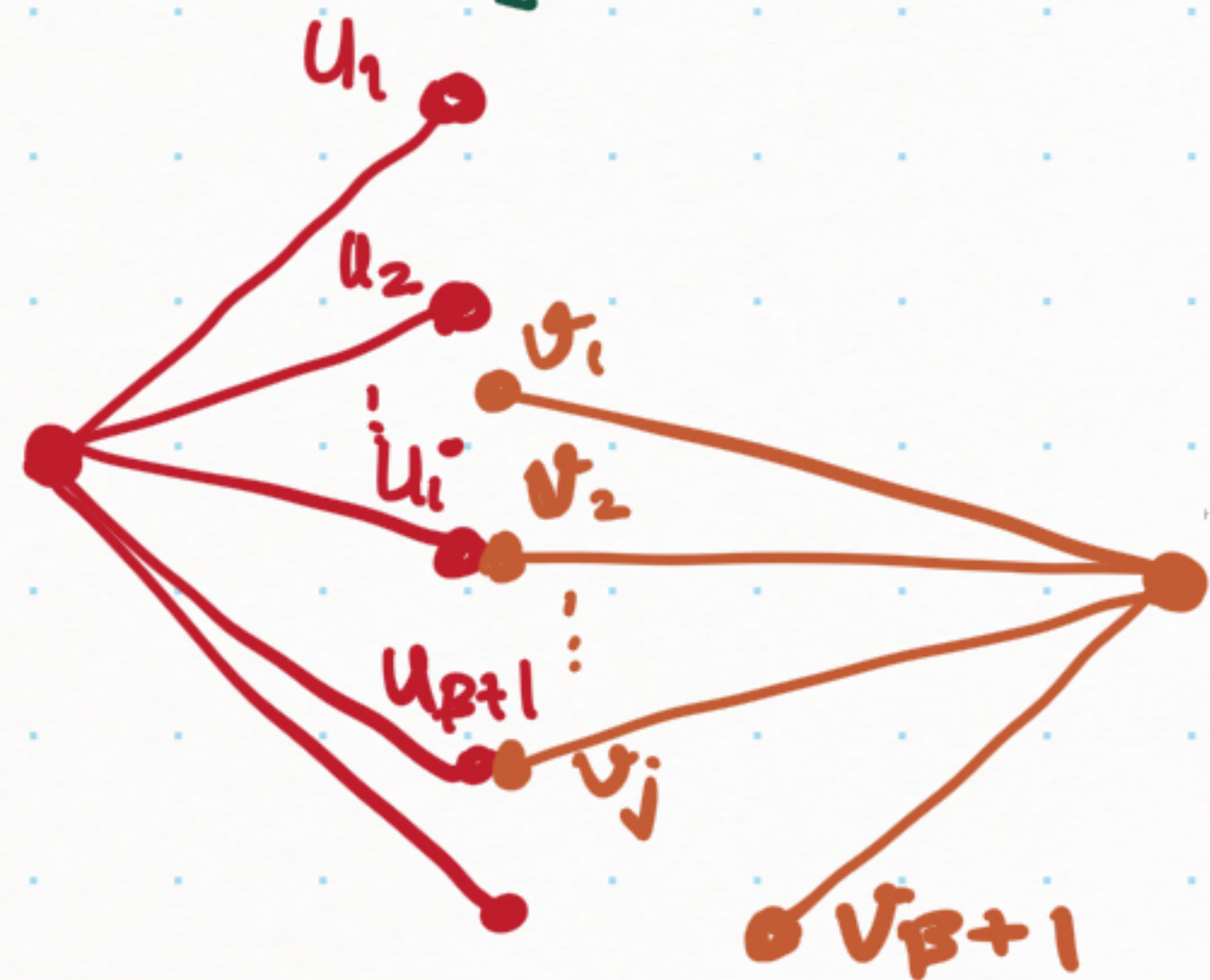
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only if $\{u_1, \dots, u_{\beta+1}\} \cap \{v_1, \dots, v_{\beta+1}\} \neq \emptyset$

$\{u_1, \dots, u_{\beta+1}\}$ is fixed & we count how
many choices we have for $\{v_1, \dots, v_{\beta+1}\}$.

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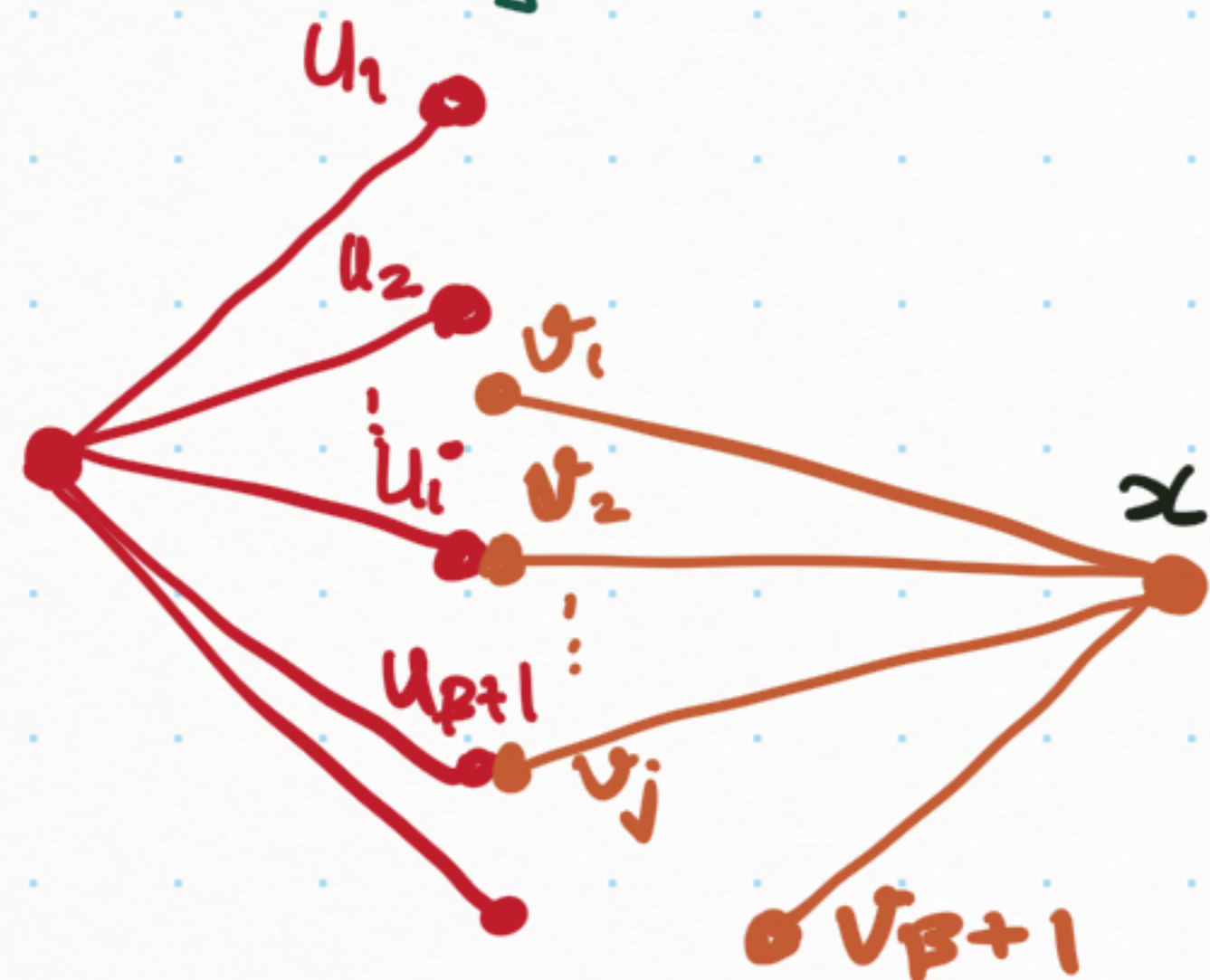
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$\rightarrow (\beta+1)$ choices among $\{u_1, \dots, u_{\beta+1}\}$ to pick a u_i in common

$\rightarrow \Delta$ choices among neighbors of u_i to pick x , common neighbor of all $\{u_1, \dots, u_{\beta+1}\}$

$\rightarrow \binom{\Delta-1}{\beta}$ choices among neighbors of x to pick the remaining β vertices of $\{u_1, \dots, u_{\beta+1}\}$.



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Apply Symmetric LLL.

Then we are forced to use
and $d = (\beta+1)\Delta \binom{\Delta-1}{\beta}$

$$p = 1/Q$$

But

$$pd \geq \frac{(\beta+1)\Delta^\beta}{\beta! 16 \Delta^{1+\beta}}$$

$$\geq \frac{\Delta^{\beta-\frac{1}{2}}}{16(\beta-1)!} \geq \frac{\beta^{\beta-1}}{16(\beta-1)!} \gg 1$$

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Apply Asymmetric LLL. Since $Q = 16\Delta^{1+\beta}$, each probability $< 1/2$

$$\begin{aligned} \sum_{j \in N(i)} P[A_j] &\leq \left[(\beta+1)\Delta \frac{1}{Q} \right] + \left[(\beta+1)\Delta \binom{\Delta-1}{\beta} \frac{1}{Q^\beta} \right] \quad \leftarrow \text{Why?} \\ &\leq (\beta+1)\Delta/Q + (\beta+1)\Delta^{\beta+1}/\beta! Q^\beta \leq \frac{(\beta+1)}{16\Delta^\beta} + \frac{(\beta+1)}{\beta! 16^\beta} \\ &\leq 1/4 \quad \text{if } \Delta > \beta^\beta \text{ and } \beta \geq 2 \end{aligned}$$