Math 554

Hemanshu Kaul

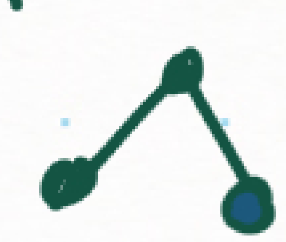
kaul@iit.edu

Setup for Lovász Local Lemma:

Defn An event B is mutually independent of events $A_1, \dots, A_k$ if $\mathbb{P}[B \mid C_{i_1} \wedge C_{i_2} \wedge \dots \wedge C_{i_\ell}] = \mathbb{P}[B]$ $\forall \ell$ $\forall 1 \le i_1 < i_2 < \dots < i_\ell \le k$.
where each C_{i_j} is either A_{i_j} or $\overline{A_{i_j}}$.

Defn G is said to be a dependency graph of events $A_1, \dots, A_n$ if $V(G) = [n]$ and for each i, A_i is mutually independent of all events A_j s.t. $ij \notin E(G)$.

Caution! mutual independence $\neq$ pairwise independence.

e.g. Pick $x_1, x_2, x_3 \in \{0,1\}$ uniformly & independently at random.
For $i = 1, 2, 3$, let A_i be event $x_{i+1} + x_{i+2} = 0 \pmod 2$ (indices mod 3)
Then these events are pairwise independent but not mutually ind.
K_3 is not a valid dep. graph, but  is a valid dep. graph.
or 

However, in many (most?) applications of LLL, the underlying probability space is a product probability space i.e., it is based on a collection of independent random experiments and each event A_i is determined by a subcollection S_i of these independent experiments.

However, in many (most?) applications of LLL, the underlying probability space is a product probability space i.e., it is based on a collection of independent random experiments and each event A_i is determined by a subcollection S_i of these independent experiments.

Mutual Independence Principle Let $Z_1, Z_2, \ldots, Z_m$ be independent experiments and $A_1, \ldots, A_n$ be events s.t. the occurence of each A_i is determined by a subset S_i of $Z_1, \ldots, Z_m$. If S_i is disjoint from $S_{j_1}, \ldots, S_{j_k}$ then A_i is mutually ind. of $\{A_{j_1}, \ldots, A_{j_k}\}$

However, in many (most?) applications of LLL, the underlying probability space is a product probability space i.e., it is based on a collection of independent random experiments and each event A_i is determined by a subcollection S_i of these independent experiments.

Mutual Independence Principle Let $Z_1, Z_2, \dots, Z_m$ be independent experiments and $A_1, \dots, A_n$ be events s.t. the occurrence of each A_i is determined by a subset S_i of $Z_1, \dots, Z_m$. If S_i is disjoint from $S_{j_1}, \dots, S_{j_k}$ then A_i is mutually ind. of $\{A_{j_1}, \dots, A_{j_k}\}$

In this set-up, a valid dependency graph can be formed by placing an edge $i \leftrightarrow j$ based on $|S_i \cap S_j|$.

e.g. in our application of LLL to $R(k,k)$, $Z_1, \dots, Z_m$ are the independent "coin flips" used to color m edges of K_n. A_S is independent of all $A_{S'}$ when S & S' share no edge, $|S \cap S'| \leq 1$

neighborhood in a dependency graph

Lovász Local Lemma, General Form

Let $A_1, \dots, A_n$ be events. For each $i \in [n]$, let $N(i) \subseteq [n]$ be such that A_i is independent of $\{A_j : j \notin \{i\} \cup N(i)\}$.

If $x_1, \dots, x_n \in [0,1)$ satisfy

$$\mathbb{P}[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j) \quad \forall i \in [n]$$

then with probability $\geq \prod_{i=1}^{n} (1 - x_i)$, none of the events A_i occur.

Lovász Local Lemma, General Form

Let $A_1, \dots, A_n$ be events. For each $i \in [n]$, let $N(i) \subseteq [n]$ be such that A_i is independent of $\{A_j : j \notin \{i\} \cup N(i)\}$.

If $x_1, \dots, x_n \in [0,1)$ satisfy

$$\mathbb{P}[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j) \quad \forall i \in [n]$$

then with probability $\geq \prod_{i=1}^{n} (1 - x_i)$, none of the events A_i occur.

LLL, Symmetric Form

Let G be a dependency graph of events $A_1, \dots, A_n$ with $\Delta(G) \leq d$ and $\mathbb{P}[A_i] \leq p \ \forall i$. Then,

$$ep(d+1) < 1 \implies \mathbb{P}\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$$

Proof Set $x_i = 1/(d+1) < 1 \ \forall i$

Then $x_i \prod_{j \in N(i)} (1 - x_j) \geq \frac{1}{d+1} \left(1 - \frac{1}{d+1}\right)^d > \frac{1}{(d+1)e} \geq p$

$\leftarrow$ neighborhood in dep. graph.

Lovász Local Lemma, General Form

Let $A_1, \dots, A_n$ be events. For each $i \in [n]$, let $N(i) \subseteq [n]$ be such that A_i is independent of $\{A_j : j \notin \{i\} \cup N(i)\}$.

If $x_1, \dots, x_n \in [0,1)$ satisfy

$$\mathbb{P}[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j) \quad \forall i \in [n]$$

then with probability $\geq \prod_{i=1}^{n} (1 - x_i)$, none of the events A_i occur.

LLL, Symmetric Form

Let G be a dependency graph of events $A_1, \dots, A_n$ with $\Delta(G) \leq d$ and $\mathbb{P}[A_i] \leq p \ \forall i$. Then,

$$ep(d+1) < 1 \Rightarrow \mathbb{P}\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$$

e is sharp; cannot be replaced by any smaller constant

often, checking $4p(d+1) < 1$ is enough.

neighborhood in a dependency graph

Lovász Local Lemma, General Form

Let $A_1, \ldots, A_n$ be events. For each $i \in [n]$, let $N(i) \subseteq [n]$ be such that A_i is independent of $\{A_j : j \notin \{i\} \cup N(i)\}$.

If $x_1, \ldots, x_n \in [0,1)$ satisfy

$$\mathbb{P}[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j) \quad \forall i \in [n]$$

then with probability $\geq \prod_{i=1}^{n} (1 - x_i)$, none of the events A_i occur.

Proof

Claim $\mathbb{P}[A_i \mid \bigwedge_{j \in S} \overline{A_j}] \leq x_i$ whenever $i \notin S \subseteq [n]$

With this claim, we can easily finish the proof:

$$\mathbb{P}[\overline{A_1} \wedge \overline{A_2} \wedge \ldots \wedge \overline{A_n}] = \mathbb{P}[\overline{A_1}]\, \mathbb{P}[\overline{A_2} \mid \overline{A_1}]\, \mathbb{P}[\overline{A_3} \mid \overline{A_1} \overline{A_2}] \ldots \mathbb{P}[\overline{A_n} \mid \overline{A_1} \ldots \overline{A_{n-1}}]$$

$$\geq \ ?$$

neighborhood in a dependency graph

Lovász Local Lemma, General Form

Let $A_1, \dots, A_n$ be events. For each $i \in [n]$, let $N(i) \subseteq [n]$ be such that A_i is independent of $\{A_j : j \notin \{i\} \cup N(i)\}$.

If $x_1, \dots, x_n \in [0,1)$ satisfy

$$\mathbb{P}[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j) \quad \forall i \in [n]$$

then with probability $\geq \prod_{i=1}^{n} (1 - x_i)$, none of the events A_i occur.

Proof

Claim $\mathbb{P}[A_i \mid \bigwedge_{j \in S} \overline{A_j}] \leq x_i$ whenever $i \notin S \subseteq [n]$

With this claim, we can easily finish the proof:

$$\mathbb{P}[\overline{A_1} \wedge \overline{A_2} \wedge \dots \wedge \overline{A_n}] = \mathbb{P}[\overline{A_1}]\, \mathbb{P}[\overline{A_2} \mid \overline{A_1}]\, \mathbb{P}[\overline{A_3} \mid \overline{A_1} \overline{A_2}] \dots \mathbb{P}[\overline{A_n} \mid \overline{A_1} \dots \overline{A_{n-1}}]$$

$$\geq (1 - x_1)(1 - x_2)(1 - x_3) \cdots (1 - x_n)$$

<u>Claim</u> $\mathbb{P}[A_i \mid \bigwedge_{j\in S} \bar{A}_j] \le x_i$ whenever $i \notin S \subseteq [n]$

Proof by induction on $|S|$. $|S|=0$ ✓

Let $i \notin S$. Denote $S_1 = S \cap N(i)$ and $S_2 = S \setminus S_1$, so

$\mathbb{P}[A_i \mid \bigwedge_{j\in S} \bar{A}_i] = \mathbb{P}[A_i \bigwedge_{j\in S_1} \bar{A}_j \mid \bigwedge_{j\in S_2} \bar{A}_j] \Big/ \mathbb{P}[\bigwedge_{j\in S_1} \bar{A}_j \mid \bigwedge_{j\in S_2} \bar{A}_j] =: x/y$

$x \le$

<u>Claim</u> $\mathbb{P}[A_i \mid \bigwedge_{j\in S} \bar{A}_j] \le x_i$ whenever $i \notin S \subseteq [n]$

Proof by induction on $|S|$. $|S|=0$ ✓

Let $i \notin S$. Denote $S_1 = S \cap N(i)$ and $S_2 = S \setminus S_1$, so

$\mathbb{P}[A_i \mid \bigwedge_{j\in S} \bar{A}_i] = \mathbb{P}[A_i \bigwedge_{j\in S_1} \bar{A}_j \mid \bigwedge_{j\in S_2} \bar{A}_j] \Big/ \mathbb{P}[\bigwedge_{j\in S_1} \bar{A}_j \mid \bigwedge_{j\in S_2} \bar{A}_j] =: x/y$

$x \le \mathbb{P}[A_i \mid \bigwedge_{j\in S_2} \bar{A}_j] =$

<u>Claim</u> $\mathbb{P}[A_i \mid \bigwedge_{j\in S} \bar{A}_j] \le x_i$ whenever $i \notin S \subseteq [n]$

Proof by induction on $|S|$. $|S|=0$ ✓

Let $i \notin S$. Denote $S_1 = S \cap N(i)$ and $S_2 = S \setminus S_1$, so

$$\mathbb{P}[A_i \mid \bigwedge_{j\in S} \bar{A}_j] = \mathbb{P}[A_i \bigwedge_{j\in S_1} \bar{A}_j \mid \bigwedge_{j\in S_2} \bar{A}_j] \Big/ \mathbb{P}[\bigwedge_{j\in S_1} \bar{A}_j \mid \bigwedge_{j\in S_2} \bar{A}_j] =: X/Y$$

$$X \le \mathbb{P}[A_i \mid \bigwedge_{j\in S_2} \bar{A}_j] = \mathbb{P}[A_i] \le x_i \prod_{j\in N(i)} (1-x_j)$$

Set $S_1 = \{j_1, \dots, j_r\}$, so

$$Y = \mathbb{P}[\bar{A}_{j_1} \mid \bigwedge_{j\in S_2} \bar{A}_j]\, \mathbb{P}[\bar{A}_{j_2} \mid \bar{A}_{j_1} \bigwedge_{j\in S_2} \bar{A}_j] \cdots \mathbb{P}[\bar{A}_{j_r} \mid \bar{A}_{j_1} \cdots \bar{A}_{j_{r-1}} \bigwedge_{j\in S_2} \bar{A}_j]$$

Claim $\mathbb{P}[A_i \mid \bigwedge_{j\in S} \bar{A}_j] \le x_i$ whenever $i \notin S \subseteq [n]$

Proof by induction on $|S|$. $|S|=0$ ✓

Let $i \notin S$. Denote $S_1 = S \cap N(i)$ and $S_2 = S \setminus S_1$, so

$$\mathbb{P}[A_i \mid \bigwedge_{j\in S} \bar{A}_i] = \mathbb{P}[A_i \bigwedge_{j\in S_1} \bar{A}_j \mid \bigwedge_{j\in S_2} \bar{A}_j] \Big/ \mathbb{P}[\bigwedge_{j\in S_1} \bar{A}_j \mid \bigwedge_{j\in S_2} \bar{A}_j] =: x/y$$

$$x \le \mathbb{P}[A_i \mid \bigwedge_{j\in S_2} \bar{A}_j] = \mathbb{P}[A_i] \le x_i \prod_{j\in N(i)} (1-x_j)$$

Set $S_1 = \{j_1, \dots, j_r\}$, so

$$y = \mathbb{P}[\bar{A}_{j_1} \mid \bigwedge_{j\in S_2} \bar{A}_j]\, \mathbb{P}[\bar{A}_{j_2} \mid \bar{A}_{j_1} \bigwedge_{j\in S_2} \bar{A}_j] \cdots \mathbb{P}[\bar{A}_{j_r} \mid \bar{A}_{j_1} \cdots \bar{A}_{j_{r-1}} \bigwedge_{j\in S_2} \bar{A}_j]$$

$$\ge (1-x_{j_1})(1-x_{j_2}) \cdots (1-x_{j_r}) \quad \text{Why?}$$

$$\ge \prod_{j\in N(i)} (1-x_j) \quad \because ?$$

Claim $\mathbb{P}[A_i \mid \bigwedge_{j\in S} \bar{A}_j] \le x_i$ whenever $i \notin S \subseteq [n]$

Proof by induction on $|S|$. $|S|=0$ ✓

Let $i \notin S$. Denote $S_1 = S \cap N(i)$ and $S_2 = S \setminus S_1$, so

$$\mathbb{P}[A_i \mid \bigwedge_{j\in S} \bar{A}_i] = \mathbb{P}[A_i \bigwedge_{j\in S_1} \bar{A}_j \mid \bigwedge_{j\in S_2} \bar{A}_j] \Big/ \mathbb{P}[\bigwedge_{j\in S_1} \bar{A}_j \mid \bigwedge_{j\in S_2} \bar{A}_j] =: x/y$$

$$x \le \mathbb{P}[A_i \mid \bigwedge_{j\in S_2} \bar{A}_j] = \mathbb{P}[A_i] \le x_i \prod_{j\in N(i)} (1-x_j)$$

Set $S_1 = \{j_1, \dots, j_r\}$, so

$$y = \mathbb{P}[\bar{A}_{j_1} \mid \bigwedge_{j\in S_2} \bar{A}_j]\, \mathbb{P}[\bar{A}_{j_2} \mid \bar{A}_{j_1} \bigwedge_{j\in S_2} \bar{A}_j] \dots\dots \mathbb{P}[\bar{A}_{j_r} \mid \bar{A}_{j_1} \cdots \bar{A}_{j_{r-1}} \bigwedge_{j\in S_2} \bar{A}_j]$$

$$\ge (1-x_{j_1})(1-x_{j_2}) \dots (1-x_{j_r})$$ Why? Induction Hypothesis

$$\ge \prod_{j\in N(i)} (1-x_j)$$ $\because S_1 \subseteq N(i)$.

■

A few comments on LLL

① Lopsided Lovasz Local Lemma

Note the following step in proof of LLL

$$X \leq \mathbb{P}[A_i \mid \bigwedge_{j \in S_2} \overline{A}_j] = \mathbb{P}[A_i] \leq x_i \prod_{j \in N(i)} (1-x_j)$$

The proof would work even if changed to "$\leq$".

i.e, we replace the independence assumption on A_i vs. non-neighb of A_i with "each A_i is non-negatively dependent on its non-neighbors"

That is, we have a negative dependency graph where intuitively (!) neighbors of each event consist of events that are negatively dependent on it.

Modify LLL to LLLL by replacing the independence condition by

$$\mathbb{P}[A_i \mid \bigwedge_{j \in S} \overline{A}_j] \leq \mathbb{P}[A_i] \quad \forall i \in [n] \text{ and } S \subseteq [n] \setminus (\{i\} \cup N(i))$$

Erdös & Spencer 1991 applied LLLL to show:

<u>Theorem</u> Every $n \times n$ array where each entry appears at most $n/4e$ times has a Latin transversal.

where,
- a <u>transversal</u> of an $n \times n$ array is a set of n entries with one in each row and column.
- a <u>Latin Transversal</u> is a transversal with distinct entries. e.g.

1	2	3
2	3	1
3	1	2

A	K	J	Q
Q	J	K	A
J	Q	A	K
K	A	Q	J

(Euler:)
$n \times n$ array with each symbol appearing exactly once in each row & column.

<u>Conjecture (Ryser 1967)</u>
Each <u>Latin square</u> of odd order has at least one transversal

Balasubramanian 1990 showed: every L.S. of even order has even # transversals.

A few comments on LLL

② Algorithmic Lovasz Local Lemma

LLL tells us a "good structure" exists but the guaranteed probability is often very small. How can we sample such a "good structure" efficiently?

A few comments on LLL

② Algorithmic Lovasz Local Lemma

LLL tells us a "good structure" exists but the guaranteed probability is often very small. How can we sample such a "good structure" efficiently?

Moser (2009) & Moser-Tardos (2010) came up with the first efficient LLL algorithm in the "Product Probability Space" setup.
↳ events are based on underlying independent random experiments

Moser-Tardos LLL algorithm

Initialize all random variables.
While "there are violated events" Do → Pick an arbitrary violated event and resample its variables.

Moser-Tardos 2010 If $\exists x_i \in [0,1)$ s.t. $\mathbb{P}[A_i] \leq x_i \prod_{j \in N(i)} (1-x_j) \ \forall i \in [n]$

The above randomized algorithm runs in expected number of steps at most $\sum_i x_i/1-x_i$.

Note that the Moser–Tardos algorithm is a <u>Las Vegas algorithm</u> → [randomized algo that always terminates sucessfully but runtime might be long]
with polynomial expected runtime

Other type of randomized algorithm is a <u>Monte Carlo Algo</u>, which always runs in bounded time but may return a bad result with some small probability.

A LV algo can be converted to a MC algo by terminating the algo after some steps (larger than expected #) & applying Markov's ineq. to bound probability of failure.

Note that the Moser–Tardos algorithm is a Las Vegas algorithm → [randomized algo that always terminates sucessfully but runtime might be long]
with polynomial expected runtime

Other type of randomized algorithm is a Monte Carlo Algo, which always runs in bounded time but may return a bad result with some small probability.
A LV algo can be converted to a MC algo by terminating the algo after some steps (larger than expected #) & applying Markov's ineq. to bound probability of failure.

Work on algorithmic LLL began with Beck in 1991 with an algorithm for finding a proper 2-coloring of a k-uniform hypergraph where each edge intersects $< 2^{k/48}$ other hyperedges.
(& since 2010, an explosion: see (Entropy Compression Method, Local Cut Lemma)

We saw earlier: every k-uniform hypergraph with $< 2^{k-1}$ edges is 2-colorable.

<u>Theorem</u> A k-uniform hypergraph is 2-colorable if every edge intersects less than $2^{k-1}/e - 1$ other edges.

<u>Proof</u> Experiment?

Bad events?

Dependency?

We saw earlier: every k-uniform hypergraph with $< 2^{k-1}$ edges is 2-colorable.

<u>Theorem</u> A k-uniform hypergraph is 2-colorable if every edge intersects less than $2^{k-1}/e - 1$ other edges.

<u>Proof</u> Experiment? Color each vertex independently & u.a.r with blue or red.

Bad events? For each edge f, $A_f \equiv$ event f is monochromatic, $\mathbb{P}[A_f] = 2^{1-k} = p$

Dependency?

We saw earlier: every k-uniform hypergraph with $< 2^{k-1}$ edges is 2-colorable.

Theorem A k-uniform hypergraph is 2-colorable if every edge intersects less than $2^{k-1}/e - 1$ other edges.

Proof Experiment? Color each vertex independently & u.a.r with blue or red.

Bad events? For each edge f, $A_f \equiv$ event f is monochromatic. $\Pr[A_f] = 2^{1-k} = p$

Dependency? By Mutual Independence Principle (MIP), each A_f is mutually independent of every $A_{f'}$ where $f \cap f' = \phi$

Since each f intersects less than $2^{k-1}/e - 1$ other f', we can pick $d < 2^{k-1}/e - 1$ i.e, $ep(d+1) < 1$ & LLL applies and $\exists$ 2-coloring of the hypergraph. ∎

In applications of LLL, even if the random process is the most obvious one, the choice of "bad events" can be subtle.

We know every graph G contains an ind. set of size $\geq \frac{|V(G)|}{\Delta(G)+1}$

Why? How?

In applications of LLL, even if the random process is the most obvious one, the choice of "bad events" can be subtle.

We know every graph G contains an ind. set of size $\geq |V(G)| / \Delta(G)+1$

Why? How?
$\sum_{i=1}^{k} (d_i+1) \geq n$
i.e. $k(\Delta(G)+1) \geq n$

By decreasing the desired size of ind. set by a constant factor $(2e)$, we can guarantee a structure on the independent set.

Lemma Let $V_1 \cup \ldots \cup V_t$ be a partition of $V(G)$ with each $|V_i| \geq 2e\Delta(G)$. Then there exists an independent set in G consisting of one vertex from each V_i.

In applications of LLL, even if the random process is the most obvious one, the choice of "bad events" can be subtle.

We know every graph G contains an ind. set of size $\geq |V(G)|/\Delta(G)+1$

Why? How? $\sum_{i=1}^{k}(d_i+1) \geq n$ i.e. $k(\Delta(G)+1) \geq n$

By decreasing the desired size of ind. set by a constant factor $(2e)$, we can guarantee a structure on the independent set.

Lemma Let $V_1 \cup \ldots \cup V_t$ be a partition of $V(G)$ with each $|V_i| \geq 2e\Delta(G)$. Then there exists an independent set in G consisting of one vertex from each V_i.

WLOG, we may assume $|V_i| = \lceil 2e\Delta(G) \rceil = a$, by removing excess vertices from each V_i.

Random Procedure? Pick $v_i \in V_i$ uniformly at random, independently to form a random set S.

Bad Events?

Attempt 1

vertex chosen from V_i

vertex chosen from V_j

$A_{i,j} \equiv$ "v_i & v_j have an edge between them"

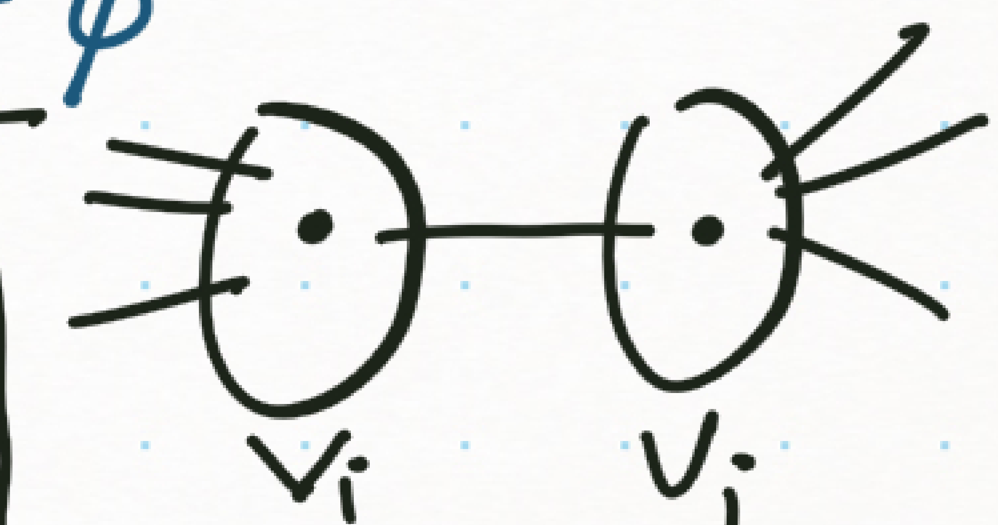
for each $1 \le i < j \le t$

$\mathbb{P}[A_{i,j}] \le \Delta(G)/a = p$

Dependency graph: $A_{i,j} \leftrightarrow A_{k,l}$ if $\overbrace{\{i,j\}\cap\{k,l\} \neq \emptyset}^{\text{common endpt between } v_iv_j \text{ \& } v_kv_l}$

max degree of Dep. graph $\le 2\Delta(G)a = d$

Then, $ep(d+1) \approx 2e(\Delta(G))^2 >> 1$



$\le \Delta(G)a$ choices from V_i & $N(V_i)$

Bad Events?

vertex chosen from V_i

vertex chosen from V_j

Attempt 1 $A_{i,j} \equiv$ "v_i & v_j have an edge between them"

for each $1 \le i < j \le t$

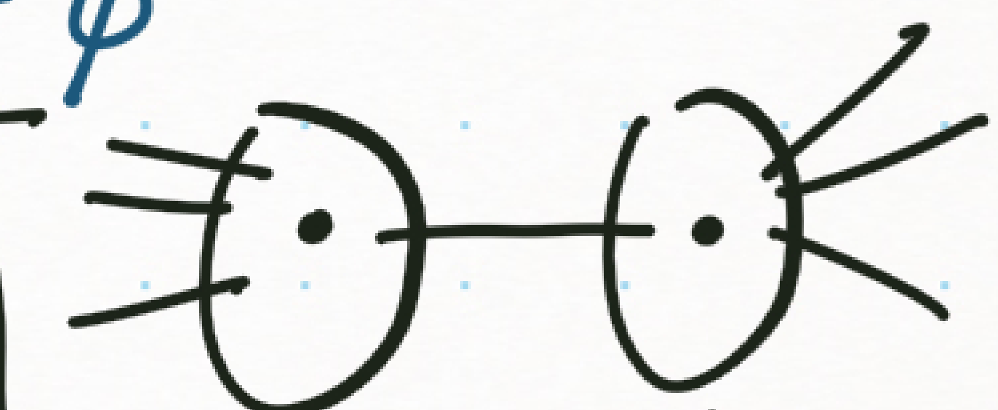
$\mathbb{P}[A_{i,j}] \le \Delta(G)/a = p$

common endpt. between v_iv_j & v_kv_ℓ

Dependency graph: $A_{i,j} \leftrightarrow A_{k,\ell}$ if $\overbrace{\{i,j\} \cap \{k,\ell\} \neq \emptyset}$

max degree of Dep. graph $\le 2\Delta(G)a = d$

Then, $ep(d+1) \approx 2e(\Delta(G))^2 >> 1$



V_i V_j

$\le \Delta(G)a$ choices from V_i & $N(V_i)$

Attempt 2 $A_e \equiv$ "both endpoints of e chosen"

for each $e \in E(G)$ with endpt.s in distinct parts.

$\mathbb{P}[A_e] =$

Bad Events?

vertex chosen from V_i — vertex chosen from V_j

Attempt 1 $\quad A_{i,j} \equiv$ "v_i & v_j have an edge between them"

for each $1 \le i < j \le t$

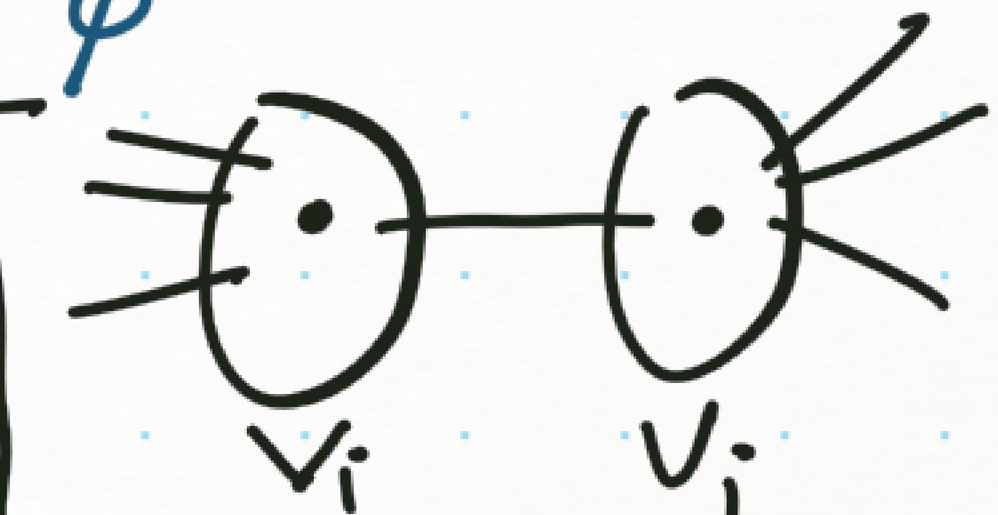
$\mathbb{P}[A_{i,j}] \le \Delta(G)/a = p$

common endpt. between v_iv_j & v_kv_ℓ

Dependency graph: $A_{i,j} \leftrightarrow A_{k,\ell}$ if $\{i,j\} \cap \{k,\ell\} \neq \emptyset$

max degree of Dep. graph $\le 2\Delta(G)a = d$

Then, $ep(d+1) \approx 2e(\Delta(G))^2 >> 1$



$\le \Delta(G)a$ choices from V_i & $N(V_i)$

Attempt 2 $\quad A_e \equiv$ "both endpoints of e chosen"

for each $e \in E(G)$ with endpt.s in distinct parts.

$\mathbb{P}[A_e] = \frac{1}{a^2} = p$

Dependency graph: By MIP, A_e (for $e = uv$) is mutually independent of the events corresponding to edges outside $V_i \cup V_j$ shares $u \in V_i$ & $v \in V_j$

$\therefore d \le 2\Delta(G)a - 1$ and $ep(d+1) \le e\frac{1}{a^2}2\Delta(G)a = \frac{2e\Delta(G)}{\lceil 2e\Delta(G)\rceil} < 1$.

<u>Thm (Reed 1999)</u> Let L be a list assignment for a graph G s.t. $|L(v)| \geq k$ for all v. If every color in the palette $\bigcup_{v \in V(G)} L(v)$ appears in at most $k/2e$ of the lists in each vertex neighborhood, then G has a proper L-coloring.

Thm (Reed 1999) Let L be a list assignment for a graph G s.t. $|L(v)| \geq k$ for all v. If every color in the palette $\bigcup_{v \in V(G)} L(v)$ appears in at most $k/2e$ of the lists in each vertex neighborhood, then G has a proper L-coloring.

Proof WLOG $|L(v)| = k$

For each $v \in V(G)$, choose a color u.a.r. from $L(v)$, independently.

Bad Events? $A_{uv} \equiv$ event both u & v receive the same color for all $uv \in E(G)$.

Check LLL fails to apply with this definition.

Thm (Reed 1999) Let L be a list assignment for a graph G s.t. $|L(v)| \geq k$ for all v. If every color in the palette $\bigcup_{v \in V(G)} L(v)$ appears in at most $k/2e$ of the lists in each vertex neighborhood, then G has a proper L-coloring.

Proof WLOG $|L(v)| = k$

For each $v \in V(G)$, choose a color u.a.r. from $L(v)$, independently

For each edge uv and colors $c \in L(u) \cap L(v)$, define

$A_{uv,c} \equiv$ event u & v both receive color c.

Thm (Reed 1999) Let L be a list assignment for a graph G s.t. $|L(v)| \geq k$ for all v. If every color in the palette $\bigcup_{v \in V(G)} L(v)$ appears in at most $k/2e$ of the lists in each vertex neighborhood, then G has a proper L-coloring.

Proof WLOG $|L(v)| = k$

For each $v \in V(G)$, choose a color u.a.r. from $L(v)$, independently.

For each edge uv and colors $c \in L(u) \cap L(v)$, define

$A_{uv,c} \equiv$ event u & v both receive color c.

$\mathbb{P}[A_{uv,c}] = \frac{1}{k^2} = p$.

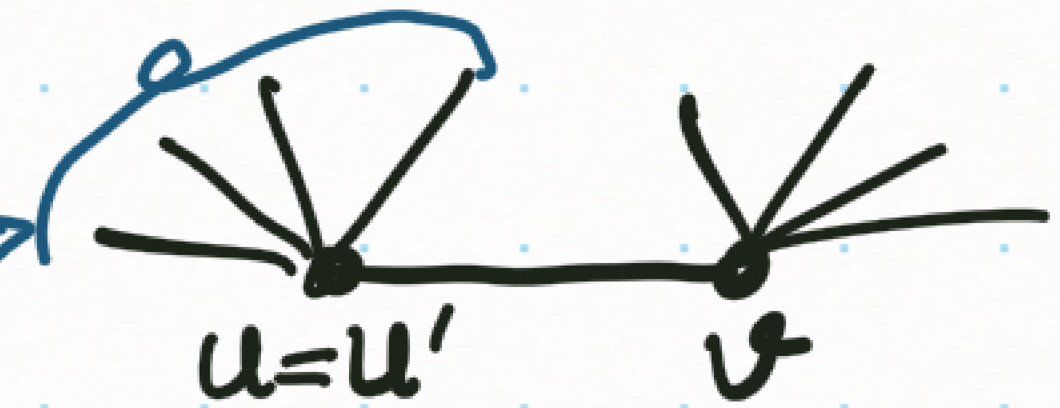
By MIP, $A_{uv,c}$ is mutually ind. of set of all such events whose edges are disjoint from u,v.

To bound d, we need to count events corresponding to edges incident to edge uv.

For $u'v'$ incident to uv, the event $A_{u'v';c'}$ is defined iff $c' \in L(u') \cap L(v')$

Suppose $u = u'$ & $v \neq v'$,

fixed $u' = u$ & $v' \in N(u') - \{v\}$

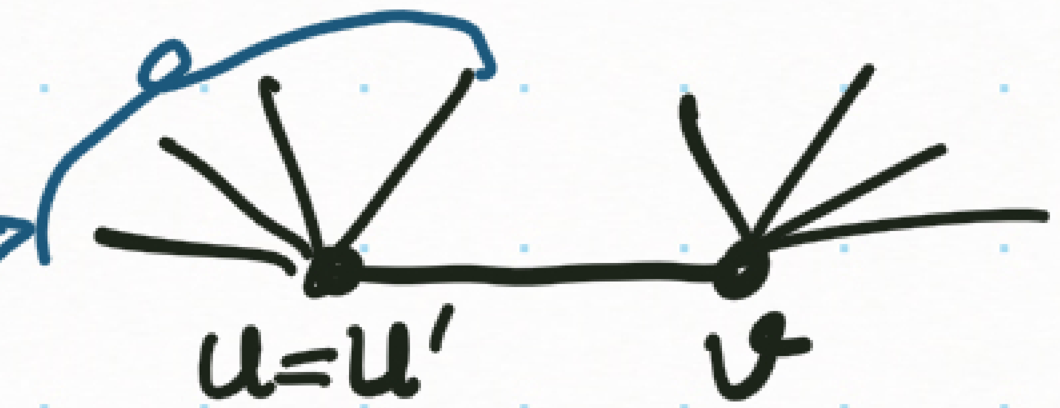


then atmost . ? such events exist for each $c' \in L(u') = L(u)$

For $u'v'$ incident to uv, the event $A_{u'v';c'}$ is defined iff $c' \in L(u') \cap L(v')$

Suppose $u = u'$ & $v \neq v'$, then atmost $\frac{k}{2e} - 1$ such events exist for each $c' \in L(u') = L(u)$ (excluding $c \in L(u) \cap L(v)$ (for the event $A_{uv,c}$))

fixed $u' = u$ & $v' \in N(u') - \{v\}$



Similarly, when $v = v'$ & $u \neq u'$

#colors in each list

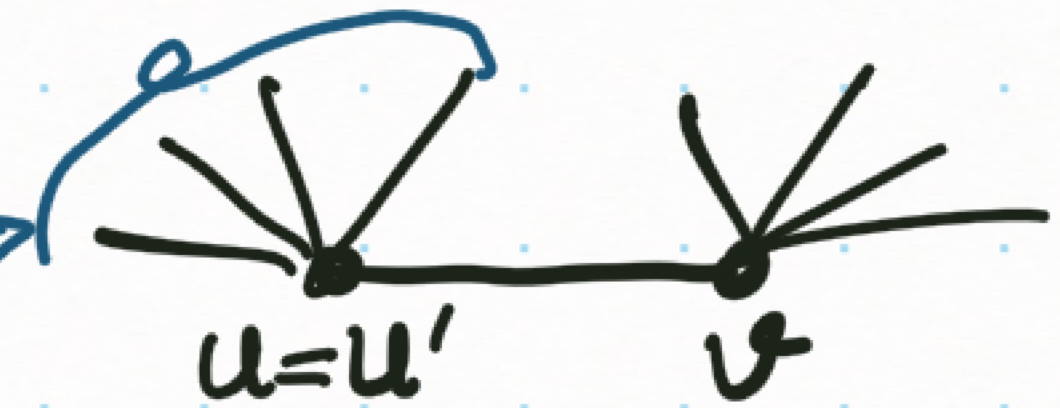
$$\therefore d \leq k \cdot 2 \cdot \left(\frac{k}{2e} - 1\right)$$

2 possibilities for $u = u'$ & $v = v'$

For $u'v'$ incident to uv, the event $A_{u'v';c'}$ is defined iff $c' \in L(u') \cap L(v')$

Suppose $u = u'$ & $v \neq v'$, then atmost $\frac{k}{2e} - 1$ such events exist for each $c' \in L(u') = L(u)$ (excluding $c \in L(u) \cap L(v)$ (for the event $A_{uv,c}$))

fixed $u' = u$ & $v' \in N(u') - \{v\}$



Similarly, when $v = v'$ & $u \neq u'$

colors in each list

$$\therefore d \leq k \cdot 2 \cdot \left(\frac{k}{2e} - 1\right)$$

2 possibilities for $u=u'$ & $v=v'$

Then, $$e\,p\,(d+1) \leq e \frac{1}{k^2}\left(2k\left(\frac{k}{2e} - 1\right) + 1\right) = 1 - \frac{2e}{k} + \frac{e}{k^2}$$

$$= 1 - \left(\frac{2ek - e}{k^2}\right) < 1$$

■

$\frac{k}{2e}$ was improved to $\frac{k}{2}$ [Haxell 2001] & $k(1 - o(1))$ [Reed-Sudakov 2002]

Reed conjectured that $(k-1)$ suffices.

Lovász Local Lemma, General Form

Let $A_1, \dots, A_n$ be events. For each $i \in [n]$, let $N(i) \subseteq [n]$ be such that A_i is independent of $\{A_j : j \notin \{i\} \cup N(i)\}$.

If $x_1, \dots, x_n \in [0,1)$ satisfy

$$\mathbb{P}[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j) \quad \forall i \in [n]$$

then with probability $\geq \prod_{i=1}^{n} (1 - x_i)$, none of the events A_i occur.

LLL, Symmetric Form

Let G be a dependency graph of events $A_1, \dots, A_n$ with $\Delta(G) \leq d$ and $\mathbb{P}[A_i] \leq p \ \forall i$. Then,

$$ep(d+1) < 1 \Rightarrow \mathbb{P}\left[\bigwedge_{i=1}^{n} \overline{A_i}\right] > 0$$

Symmetric LLL can be useless when there are different kinds of "bad" events with varying probabilities.

Lovász Local Lemma, General Form

Let $A_1, \dots, A_n$ be events. For each $i \in [n]$, let $N(i) \subseteq [n]$ be such that A_i is independent of $\{A_j : j \notin \{i\} \cup N(i)\}$.

If $x_1, \dots, x_n \in [0,1)$ satisfy

$$\mathbb{P}[A_i] \leq x_i \prod_{j \in N(i)} (1 - x_j) \quad \forall i \in [n]$$

then with probability $\geq \prod_{i=1}^{n} (1 - x_i)$, none of the events A_i occur.

LLL, Asymmetric Form (aka Neighborhood Local Lemma)

Let $A_1, \dots, A_n$ be events. Let $N(i)$ be the neighborhood of A_i in a dependency graph, as defined above.

If $\mathbb{P}[A_i] < \frac{1}{2}$ and $\sum_{j \in N(i)} \mathbb{P}[A_j] \leq 1/4$ $\forall i$ then $\mathbb{P}[\bigwedge_{i=1}^{n} \overline{A_i}] > 0$.

Proof: Set $x_i = 2\mathbb{P}[A_i]$ $\forall i$ in LLLG. (so $x_i \in [0,1)$)

Then, $x_i \prod_{j \in N(i)} (1 - x_j) \geq x_i \left(1 - \sum_{j \in N(i)} x_j\right) = 2\mathbb{P}[A_i]\left(1 - \sum_{j \in N(i)} 2\mathbb{P}[A_j]\right) \geq \mathbb{P}[A_i]$

why? why?

Lovász Local Lemma, General Form

Let $A_1, \dots, A_n$ be events. For each $i \in [n]$, let $N(i) \subseteq [n]$ be such that A_i is independent of $\{A_j : j \notin \{i\} \cup N(i)\}$.

If $x_1, \dots, x_n \in [0,1)$ satisfy

$$\mathbb{P}[A_i] \le x_i \prod_{j \in N(i)} (1 - x_j) \quad \forall i \in [n]$$

then with probability $\ge \prod_{i=1}^{n} (1 - x_i)$, none of the events A_i occur.

LLL, Asymmetric Form (aka Neighborhood Local Lemma)

Let $A_1, \dots, A_n$ be events. Let $N(i)$ be the neighborhood of A_i in a dependency graph, as defined above.

If $\mathbb{P}[A_i] < \frac{1}{2}$ and $\sum_{j \in N(i)} \mathbb{P}[A_j] \le 1/4$ $\forall i$ then $\mathbb{P}[\bigwedge_{i=1}^{n} \overline{A_i}] > 0$.

More than one type of "bad" event with different probabilities, But few high probability events in each neighborhood $N(i)$.

LLL, Assymetric form (aka Neighborhood Local Lemma)

Let $A_1, .., A_n$ be events. Let $N(i)$ be neighborhood of A_i in a dependency graph, as defined in LLLG.

If $\mathbb{P}[A_i] < 1/2$ and $\sum_{j \in N(i)} \mathbb{P}[A_j] \leq 1/4$ $\forall i$, then $\mathbb{P}[\bigwedge_{i=1}^{n} \bar{A}_i] > 0$

Apply LLL (Symmetric) & LLL (Assymetric) to the following:

Let H be a hypergraph with $|e| \geq 3$ $\forall e \in E(H)$.

If $\sum_{\substack{f \in E(H) - e \\ \text{s.t. } e \cap f \neq \phi}} 2^{-|f|} \leq 1/8$ $\forall e \in E(H)$, then H is 2-colorable.

LLL, Assymetric form (aka Neighborhood Local Lemma)

Let $A_1, \ldots, A_n$ be events. Let $N(i)$ be neighborhood of A_i in a dependency graph, as defined in LLLG.

If $\mathbb{P}[A_i] < 1/2$ and $\sum_{j \in N(i)} \mathbb{P}[A_j] \leq 1/4$ $\forall i$, then $\mathbb{P}[\bigwedge_{i=1}^{n} \bar{A_i}] > 0$

Apply LLL (Symmetric) & LLL (Assymetric) to the following:

Let H be a hypergraph with $|e| \geq 3$ $\forall e \in E(H)$.

If $\sum_{\substack{f \in E(H) \setminus e \\ \text{s.t. } e \cap f \neq \emptyset}} 2^{-|f|} \leq 1/8$ $\forall e \in E(H)$, then H is 2-colorable.

2-color vertices uniformly at random, independently.

$A_e \equiv$ event e is monochromatic

$\mathbb{P}[A_e] = 2^{-|e|+1} \leq 2^{-3+1} = 1/4$; $\sum_{\substack{f \in E(H) \setminus e \\ e \cap f \neq \emptyset}} \mathbb{P}[A_f] = \sum_{''} 2^{-|f|+1} = 2 \sum_{''} 2^{-|f|} \leq \frac{1}{4}$ $\forall e$

LLL, Assymetric form (aka Neighborhood Local Lemma)

Let $A_1, \ldots, A_n$ be events. Let $N(i)$ be neighborhood of A_i in a dependency graph, as defined in LLLG.

If $\mathbb{P}[A_i] < 1/2$ and $\sum_{j \in N(i)} \mathbb{P}[A_j] \leq 1/4 \ \forall i$, then $\mathbb{P}[\bigwedge_{i=1}^{n} \bar{A}_i] > 0$

Defn A proper coloring of a graph G is called $\underline{\beta\text{-frugal}}$ if no color appears more than β times in the neighborhood of any vertex.

What is 1-frugal coloring of G?

LLL, Assymetric form (aka Neighborhood Local Lemma)

Let $A_1, \ldots, A_n$ be events. Let $N(i)$ be neighborhood of A_i in a dependency graph, as defined in LLL G.
If $\mathbb{P}[A_i] < 1/2$ and $\sum_{j \in N(i)} \mathbb{P}[A_j] \leq 1/4$ $\forall i$, then $\mathbb{P}[\bigwedge_{i=1}^{n} \bar{A}_i] > 0$

<u>Defn</u> A proper coloring of a graph G is called <u>β-frugal</u> if no color appears more than β times in the neighborhood of any vertex.

What is 1-frugal coloring of G?

No color is repeated in $N(v)$ $\forall v$, i.e. proper coloring if each neighborhood is converted into a clique.

That is, G has 1-frugal coloring using k colors $\iff$ G^2 has a proper coloring using k colors

How many colors does a β-frugal coloring need?

Theorem [Hind, Molloy, Reed 1997]

If G has max degree $\Delta \geq \beta^{\beta}$, then G has a β-frugal coloring with at most $16\,\Delta^{1+1/\beta}$ colors.

Theorem [Hind, Molloy, Reed 1997]

If G has max degree $\Delta \geq \beta^{\beta}$, then G has a β-frugal coloring with atmost $16\,\Delta^{1+1/\beta}$ colors.

Proof $\beta = 1$ G has 1-frugal coloring if G^2 has a proper coloring.

By Brooks, G^2 can be proper coloring with $\Delta(G^2)+1$ colors

i.e., atmost ? colors

Theorem [Hind, Molloy, Reed 1997]

If G has max degree $\Delta \geq \beta^{\beta}$, then G has a β-frugal coloring with atmost $16\,\Delta^{1+1/\beta}$ colors.

Proof $\beta = 1$ G has 1-frugal coloring if G^2 has a proper coloring.

By Brooks, G^2 can be proper coloring with $\Delta(G^2)+1$ colors

i.e., atmost Δ^2+1 colors

Note $16\,\Delta^{1+1/\beta} = 16\,\Delta^2 \geq \Delta^2+1$.

Assume $\beta \geq 2$ Assign a color to each vertex u.a.r from $Q = 16\,\Delta^{1+1/\beta}$ colors, independently.

We have two types of bad events $\longrightarrow$?

$\longrightarrow$?

Theorem [Hind, Molloy, Reed 1997]

If G has max degree $\Delta \geq \beta^{\beta}$, then G has a β-frugal coloring with atmost $16\,\Delta^{1+1/\beta}$ colors.

Proof $\beta=1$ G has 1-frugal coloring if G^2 has a proper coloring.

By Brooks, G^2 can be proper coloring with $\Delta(G^2)+1$ colors

i.e., atmost Δ^2+1 colors

Note $16\,\Delta^{1+1/\beta} = 16\,\Delta^2 \geq \Delta^2+1$.

Assume $\beta \geq 2$ Assign a color to each vertex u.a.r from $Q = 16\,\Delta^{1+1/\beta}$ colors, independently.

We have two types of bad events $\longrightarrow$ not proper coloring

$\longrightarrow$ not β-frugal

& Each type has different probabilities and different dependencies.

Type-1 events $A_{uv} \equiv u, v$ have same color, for each $uv \in E(G)$

Type-2 events $B_{u_1,\dots,u_{\beta+1}} \equiv u_1, \dots, u_{\beta+1}$ have same color, for each set of $\beta+1$ neighbors, $u_1, \dots, u_{\beta+1}$, of some vertex.

For any k, probability of k vertices having same color is $1/Q^{k-1}$, so

$\mathbb{P}[A_{uv}] = 1/Q$ and $\mathbb{P}[B_{u_1,\dots,u_{\beta+1}}] = 1/Q^{\beta}$

What about dependencies?

Type-1 events $A_{uv} \equiv u, v$ have same color, for each $uv \in E(G)$

Type-2 events $B_{u_1,\dots,u_{\beta+1}} \equiv u_1, \dots, u_{\beta+1}$ have same color, for each set of $\beta+1$ neighbors, $u_1, \dots, u_{\beta+1}$, of some vertex.

For any k, probability of k vertices having same color is $1/Q^{k-1}$, so

$\mathbb{P}[A_{uv}] = 1/Q$ and $\mathbb{P}[B_{u_1,\dots,u_{\beta+1}}] = 1/Q^{\beta}$

Each Type-1 event depends on atmost 2Δ Type-1 events and Type-2 events.

by MIP, atmost $(\Delta-1)+(\Delta-1)$ such events

Type-1 events $A_{uv} \equiv u, v$ have same color, for each $uv \in E(G)$

Type-2 events $B_{u_1,\dots,u_{\beta+1}} \equiv u_1, \dots, u_{\beta+1}$ have same color, for each set of $\beta+1$ neighbors, $u_1, \dots, u_{\beta+1}$, of some vertex.

For any k, probability of k vertices having same color is $1/Q^{k-1}$, so

$\mathbb{P}[A_{uv}] = 1/Q$ and $\mathbb{P}[B_{u_1,\dots,u_{\beta+1}}] = 1/Q^{\beta}$

Each Type-1 event depends on atmost 2Δ Type-1 events and $2\Delta\binom{\Delta-1}{\beta}$ Type-2 events.

by MIP, atmost $(\Delta-1)+(\Delta-1)$ such events

By MIP, $B_{u_1,\dots,u_{\beta+1}}$ depends on A_{uv} only if $\{u_1,\dots,u_{\beta+1}\} \cap \{u,v\} \neq \phi$

2 choices (u or v) for common vertex ($u_i = u$ or v)

Δ choices among neighbors of common vertex (say u_i) for x

$\binom{\Delta-1}{\beta}$ choices among remaining neighbors of x for the remaining $\{u_1, \dots, u_{\beta+1}\}$

Type-1 events $A_{uv} \equiv u, v$ have same color, for each $uv \in E(G)$

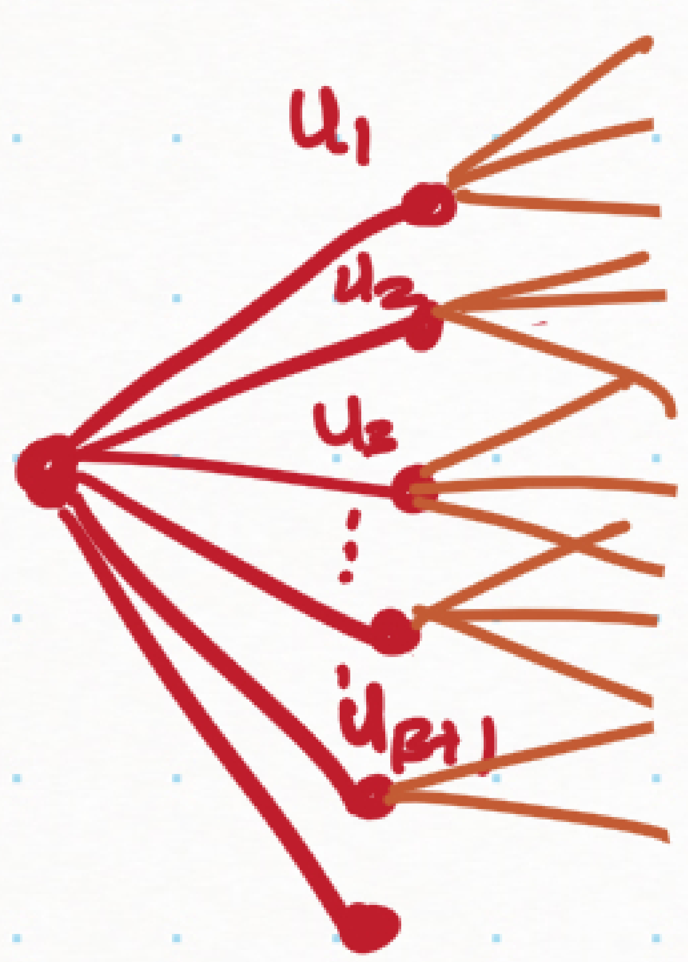
Type-2 events $B_{u_1,\dots,u_{\beta+1}} \equiv u_1, \dots, u_{\beta+1}$ have same color, for each set of $\beta+1$ neighbors, $u_1, \dots, u_{\beta+1}$, of some vertex.

For any k, probability of k vertices having same color is $1/Q^{k-1}$, so

$\mathbb{P}[A_{uv}] = 1/Q$ and $\mathbb{P}[B_{u_1,\dots,u_{\beta+1}}] = 1/Q^{\beta}$

Each Type-1 event depends on atmost 2Δ Type-1 events and $2\Delta\binom{\Delta-1}{\beta}$ Type-2 events.

Each Type-2 event depends on atmost $(\beta+1)\Delta$ Type-1 events and $(\beta+1)\Delta\binom{\Delta-1}{\beta}$ Type-2 events



at most $(\beta+1)(\Delta-1)$ edges (to use as uv in type-1 event) incident to $u_1, \dots, u_{\beta+1}$

<u>Type-1 events</u> $A_{uv} \equiv u, v$ have same color, for each $uv \in E(G)$

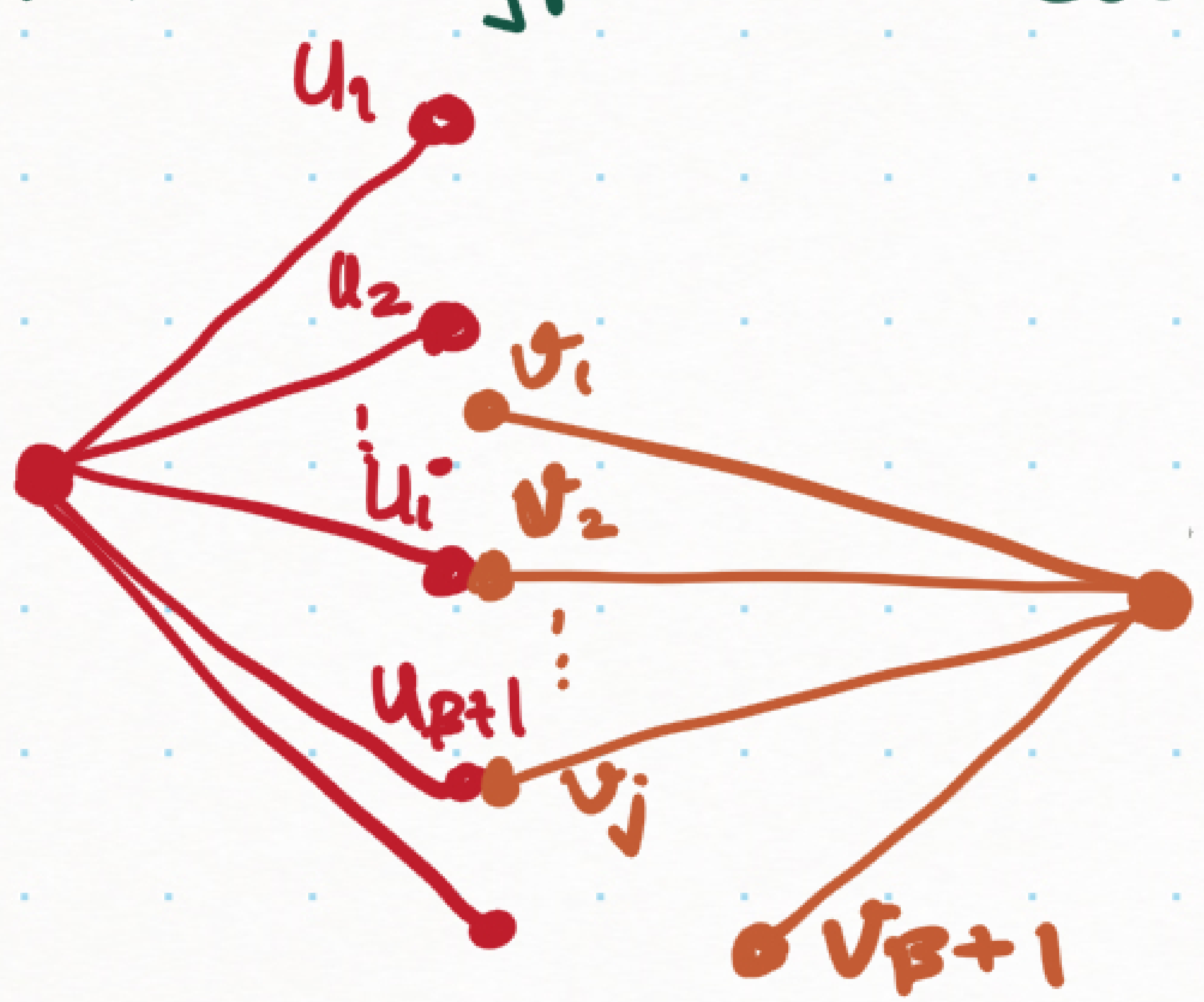
<u>Type-2 events</u> $B_{u_1,\dots,u_{\beta+1}} \equiv u_1, \dots, u_{\beta+1}$ have same color, for each set of $\beta+1$ neighbors, $u_1, \dots, u_{\beta+1}$, of some vertex.

For any k, probability of k vertices having same color is $1/Q^{k-1}$, so

$\mathbb{P}[A_{uv}] = 1/Q$ and $\mathbb{P}[B_{u_1,\dots,u_{\beta+1}}] = 1/Q^{\beta}$

Each Type-1 event depends on atmost 2Δ Type-1 events and $2\Delta\binom{\Delta-1}{\beta}$ Type-2 events.

Each Type-2 event depends on atmost $(\beta+1)\Delta$ Type-1 events and $(\beta+1)\Delta\binom{\Delta-1}{\beta}$ Type-2 events



By MIP, $B_{u_1,\dots,u_{\beta+1}}$ and $B_{v_1,\dots,v_{\beta+1}}$ are dependent only if $\{u_1, \dots, u_{\beta+1}\} \cap \{v_1, \dots, v_{\beta+1}\} \neq \emptyset$

$\{u_1, \dots, u_{\beta+1}\}$ is fixed & we count how many choices we have for $\{v_1, \dots, v_{\beta+1}\}$.

Type-1 events $A_{uv} \equiv u, v$ have same color, for each $uv \in E(G)$

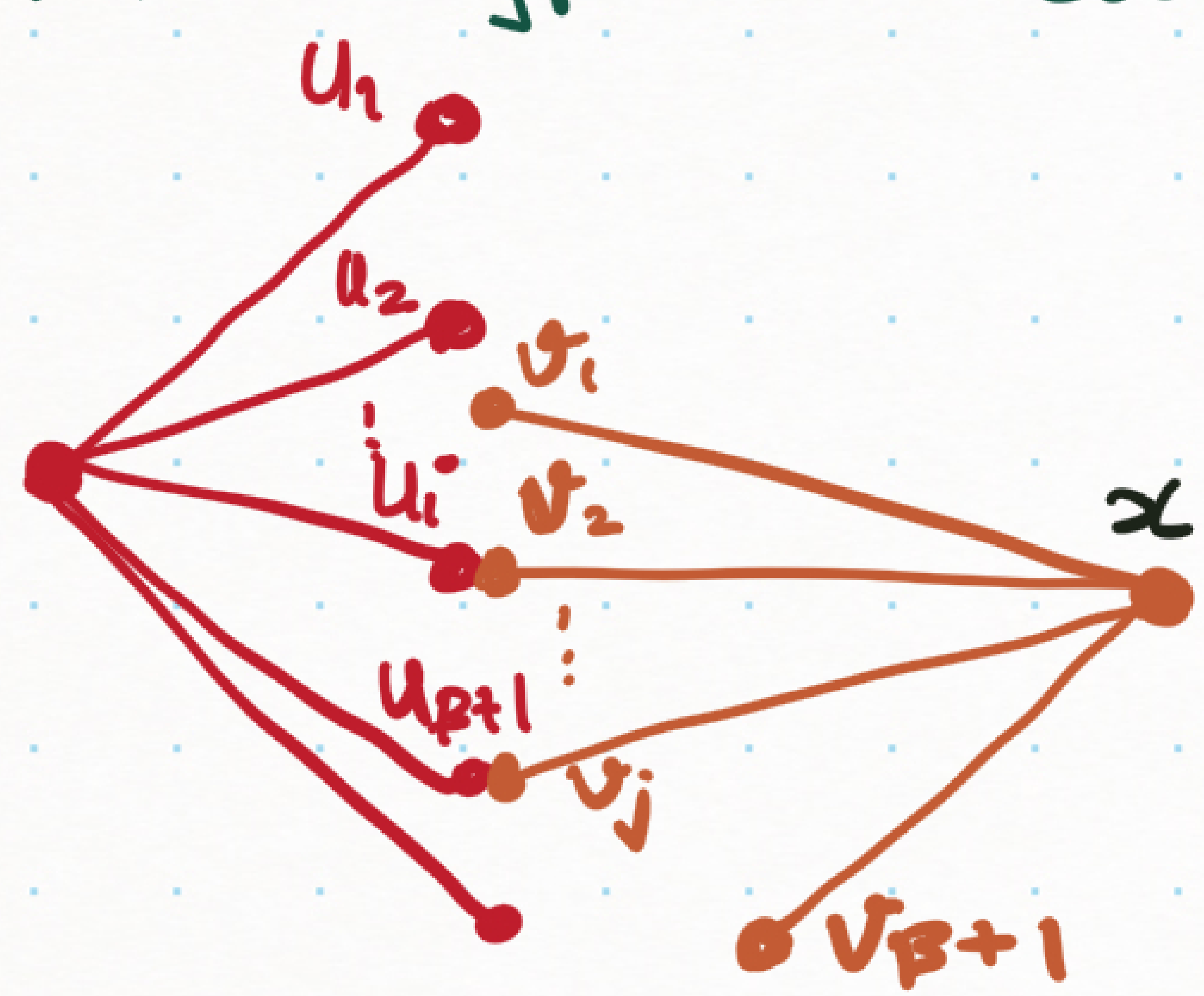
Type-2 events $B_{u_1,\dots,u_{\beta+1}} \equiv u_1, \dots, u_{\beta+1}$ have same color, for each set of $\beta+1$ neighbors, $u_1, \dots, u_{\beta+1}$, of some vertex.

For any k, probability of k vertices having same color is $1/Q^{k-1}$, so

$\mathbb{P}[A_{uv}] = 1/Q$ and $\mathbb{P}[B_{u_1,\dots,u_{\beta+1}}] = 1/Q^{\beta}$

Each Type-1 event depends on atmost 2Δ Type-1 events and $2\Delta\binom{\Delta-1}{\beta}$ Type-2 events.

Each Type-2 event depends on atmost $(\beta+1)\Delta$ Type-1 events and $(\beta+1)\Delta\binom{\Delta-1}{\beta}$ Type-2 events



$\rightarrow (\beta+1)$ choices among $\{u_1, \dots, u_{\beta+1}\}$ to pick a v_i in common

$\rightarrow \Delta$ choices among neighbors of v_i to pick x, common neighbor of all $\{v_1, \dots, v_{\beta+1}\}$

$\rightarrow \binom{\Delta-1}{\beta}$ choices among neighbors of x to pick the remaining β vertices of $\{v_1, \dots, v_{\beta+1}\}$.

Type-1 events $A_{uv} \equiv u, v$ have same color, for each $uv \in E(G)$

Type-2 events $B_{u_1,\ldots,u_{\beta+1}} \equiv u_1, \ldots, u_{\beta+1}$ have same color, for each set of $\beta+1$ neighbors, $u_1, \ldots, u_{\beta+1}$, of some vertex.

For any k, probability of k vertices having same color is $1/Q^{k-1}$, so

$\mathbb{P}[A_{uv}] = 1/Q$ and $\mathbb{P}[B_{u_1,\ldots,u_{\beta+1}}] = 1/Q^{\beta}$

Each Type-1 event depends on atmost 2Δ Type-1 events and $2\Delta\binom{\Delta-1}{\beta}$ Type-2 events.

Each Type-2 event depends on atmost $(\beta+1)\Delta$ Type-1 events and $(\beta+1)\Delta\binom{\Delta-1}{\beta}$ Type-2 events

Apply Symmetric LLL.

Then we are forced to use $p = 1/Q$ and $d = (\beta+1)\Delta\binom{\Delta}{\beta}$

But $pd \geq \frac{(\beta+1)\Delta^{\beta+1}}{\beta!\,16\Delta^{1+1/\beta}} \geq \frac{\Delta^{\beta-1/\beta}}{16(\beta-1)!} \geq \frac{\beta^{\beta^2-1}}{16(\beta-1)!} \gg 1$

Type-1 events $A_{uv} \equiv u,v$ have same color, for each $uv \in E(G)$

Type-2 events $B_{u_1,\dots,u_{\beta+1}} \equiv u_1,\dots,u_{\beta+1}$ have same color, for each set of $\beta+1$ neighbors, $u_1,\dots,u_{\beta+1}$, of some vertex.

For any k, probability of k vertices having same color is $1/Q^{k-1}$, so

$\mathbb{P}[A_{uv}] = 1/Q$ and $\mathbb{P}[B_{u_1,\dots,u_{\beta+1}}] = 1/Q^{\beta}$ $\longrightarrow (*)$

Each Type-1 event depends on atmost 2Δ Type-1 events and $2\Delta\binom{\Delta-1}{\beta}$ Type-2 events.

Each Type-2 event depends on atmost $(\beta+1)\Delta$ Type-1 events and $(\beta+1)\Delta\binom{\Delta-1}{\beta}$ Type-2 events

Apply Asymmetric LLL. Since $Q = 16\Delta^{1+1/\beta}$, each probability $< 1/2$

$$\sum_{j \in N(i)} \mathbb{P}[A_j] \le \left[(\beta+1)\Delta \frac{1}{Q}\right] + \left[(\beta+1)\Delta \binom{\Delta}{\beta} \frac{1}{Q^{\beta}}\right] \longleftarrow \text{Why?}$$

$$\le (\beta+1)\Delta/Q + (\beta+1)\Delta^{\beta+1}/\beta!\,Q^{\beta} \le (\beta+1)/16\Delta^{1/\beta} + (\beta+1)/\beta!\,16^{\beta}$$

$$\le 1/4 \quad \text{if } \Delta > \beta^{\beta} \text{ and } \beta \ge 2$$