

# Handout (section 6.2)

## ① Illustration of Proof of MIF

$$n=10: \sum_{d|10} \left( \sum_{c|(\frac{10}{d})} \mu(d) f(c) \right)$$

$$= \mu(1)[f(1)+f(2)+f(5)+f(10)] + \mu(2)[f(1)+f(5)] + \mu(5)[f(1)+f(2)] + \mu(10) f(1)$$

rearranging terms w.r.t.  $f$

$$= f(1) [\mu(1)+\mu(2)+\mu(5)+\mu(10)] + f(2) [\mu(2)+\mu(5)] + f(5) [\mu(1)+\mu(2)] + f(10) \mu(1)$$

$$= \sum_{d|10} \left( \sum_{c|(\frac{10}{d})} \mu(d) f(c) \right)$$

→ How about quotient  $\frac{f}{g}$  (given  $g(n) \neq 0 \forall n$ )?

## ★ ② Product of two multiplicative functions is also multiplicative:

Let  $f$  &  $g$  be multiplicative. To show:  $fg$  is also multiplicative

Given  $m, n$  with  $\gcd(m, n) = 1$ ,

$$\begin{aligned} (fg)(mn) &= f(mn)g(mn) = f(m)f(n)g(m)g(n) = (f(m)g(m))(f(n)g(n)) \\ &= (fg)(m) (fg)(n) \quad \text{done} \end{aligned}$$

## ③ For any $n \geq 1$ , $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0$

Proof.

Of any four consecutive integers  $n, n+1, n+2, n+3$ , one will be divisible by 4 ( $=2^2$ ), hence not square free, so will have  $\mu$ -value of 0.

## ④ For $n \geq 3$ , $\sum_{k=1}^n \mu(k!) = 1$

$$\begin{aligned} \sum_{k=1}^n \mu(k!) &= \mu(1!) + \mu(2!) + \mu(3!) + \mu(4!) + \dots + \mu(n!) \\ &= \mu(1) + \mu(2) + \mu(6) + \dots + \mu(n!) \end{aligned}$$

each of these is divisible by 4 ( $=2^2$ ) hence  $\mu$ -value is 0

$$= 1 + (-1) + 1 = 1$$

Important ★

## ⑤ If $f$ is mult., then prove that $F(n) = \sum_{d|n} \mu(d) f(d) = (1-f(p_1))(1-f(p_2)) \dots (1-f(p_r))$

where  $n = p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$  is the prime factorization of  $n$ .

Proof: Since both  $\mu$  &  $f$  are mult.,  $\mu f$  is also mult.

Applying Thm 6.4,  $F$  is also multiplicative (so its enough to find  $F(p^k)$ ,  $p$  prime)

$$\begin{aligned} F(p^k) &= \mu(1)f(1) + \mu(p)f(p) + \dots + \mu(p^k)f(p^k) = \mu(1)f(1) + \mu(p)f(p) \\ &= 1 - f(p) \quad (\because f(1)=1 \text{ as a mult. ftn.}) \end{aligned}$$

(all other  $\mu$ -values are 0)

$$\text{So, } F(n) = F(p_1^{k_1}) F(p_2^{k_2}) \dots F(p_r^{k_r})$$

$$\text{using } \otimes = (1-f(p_1))(1-f(p_2)) \dots (1-f(p_r)) \quad \text{done.}$$