Some Examples (Section 5.2)

① Show that $13 \mid 1+11^{12n+6}$ for all $n \geq 1$

Modulo 13, $1+11^{12n+6} = 1 + (11^{12})^n (11)^6$
$\equiv 1 + (1)^n (11)^6$
$\equiv 1 + (-2)^6$
$\equiv 0 \pmod{13}$

since by Fermat's thm., $11^{12} \equiv 1 \pmod{13}$

② By F.l.Thm, $a^5 \equiv a \pmod 5$
so, $a^{21} = a(a^5)^4 \equiv a(a)^4 \equiv a^5 \equiv a \pmod 5$

③ Show that $a^7 \equiv a \pmod{42}$

Note $42 = 2 \cdot 3 \cdot 7$

By F.l.T., $a^2 \equiv a \pmod 2$, $a^3 \equiv a \pmod 3$, $a^7 \equiv a \pmod 7$

Hence
$a^7 = a(a^2)^3 \equiv a(a)^3 \equiv a^4 \equiv a^2 \equiv a \pmod 2$
$a^7 = a(a^3)^2 \equiv a(a)^2 \equiv a^3 \equiv a \pmod 3$
$a^7 \equiv a \pmod 7$
imply that $a^7 \equiv a \pmod{42}$

④ If $7 \nmid a$ then prove either $7 \mid a^3-1$ or $7 \mid a^3+1$

Since $7 \nmid a$, $\gcd(a, 7) = 1$ (7 is a prime)
By F.l.T., $a^6 \equiv 1 \pmod 7$
$\Rightarrow 7 \mid a^6 - 1$
$\Rightarrow 7 \mid (a^3-1)(a^3+1)$
$\Rightarrow 7 \mid a^3-1$ or $7 \mid a^3+1$ ($\because$ 7 is prime)