MATH 553 : Homework #1 [Double 2-week HW]

Due Thursday, 9/6, in class before the lecture starts.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with it, don't hesitate to ask. You can contact me during office hours, or through email.

All problems require explicit and detailed proofs/ arguments/ reasons. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

You are allowed to discuss the homework problems with no one except your classmates, and the instructor. However, the solutions should be written by you and you alone in your own words. Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

If you discuss the problems with anyone, please note their name at the top of your HW submission under a subtitle "Collaborator" or "Discussed with:".

This is a double HW with double time allocation. Some of the topics will be discussed in class next week.

Submit a total of 9 problems out of the 10 listed below.

1. Let $\Delta(G)$ denote the maximum degree, and $\delta(G)$ denote the minimum degree of a vertex in a graph G.

Given a graph G and a subgraph H of G, consider the following two statements:

(1) $\delta(G) \geq \delta(H)$ and (2) $\Delta(G) \geq \Delta(H)$.

(a) Only one of these statements is true for all graphs. Which is it? Given a short proof of the truth of that statement.

(b) Find a counterexample showing the other statement is not always true.

2. Let $N(n, k)$ denote the number of nonisomorphic simple graphs with n vertices and k edges. Compute $N(4, 3)$ by explicitly finding all such graphs.

Find all nonisomorphic simple graphs of order 4, by finding all $N(4, k)$ as above.

Comment: Remember that you have justify your claims and explain how you have considered all possibilities for graphs on 4 vertices.

3. Solve both of following two short questions. Cut-vertex is separating set consisting of just one vertex.

(a) Let G be connected graph. Prove that: A vertex w in G is a cut-vertex if and only if there exist two vertices u and v in G such that every path between u and v passes through w.

(b) Show that every graph (containing at least one edge) has at least two vertices that are not cut-vertices.

4. If G_1 and G_2 are complementary graphs (that is, they are complements of each other, $\overline{G_1} = G_2$) then prove that at least one of them must be connected.

5. Prove that every set of six people contains at least three mutual acquaintances or three mutual strangers. As the first step, first show how to convert this statement into a graph-theoretic statement, and then prove that graph-theoretic statement.

6. Prove that a self-complementary graph with n vertices exists if and only if n or $n-1$ is divisible by 4. Recall that G is self-complementary means that G is isomorphic to its own complement, $\overline{G} = G$.

Comment: Note that this is an 'if and only if' statement which means you have to prove two implications - forward and backward. The forward implication asks you to prove that a self-complementary graph can only have a certain number of vertices. The backward implication is asking you to give a construction of a self-complementary graph on n vertices when n satisfies the divisibility condition. You have describe (use figures if you like) how such a graph on any number of vertices (not just specific small values of n) can be constructed.

7. Let G be a graph with girth 4 in which every vertex has degree k. Give a direct argument that G has at least $2k$ vertices. Determine all such graphs with exactly $2k$ vertices. (Girth means the length of the shortest cycle in a graph.)

8. Let G be a connected graph that does not have P_4 or C_4 as an induced subgraph. Prove that G has a vertex adjacent to all other vertices.

9. (a) Let G have minimum degree $\delta(G) \geq 2$. Prove that G has a cycle of length at least $\delta(G)+1$.

(b) Let G have minimum degree 3. Prove that G has a cycle of even length.

10. Let G be a graph on n vertices, $n \geq 2$. Determine the **maximum** possible number of edges in G under each of the following conditions:

a) G has an independent set of size a.

b) G has exactly k components.

Comment: Since the problem is asking you to "determine the maximum" you have to prove this in two steps. First, establish an upper bound on the number edges in such a type of graph. Second, explicitly construct such a type of graph on n vertices which has number of edges exactly equal to the bound you proved. This shows that the upper bound you proved is the best possible, thus determining the maximum.