MATH 553 : Homework #2

Due Thursday, 9/20, in class before the lecture starts.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with it, don't hesitate to ask. You can contact me during office hours, or through email, or through discussion forum in Piazza.

All problems require explicit and detailed proofs/ arguments/ reasons. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

You are allowed to discuss the homework problems with no one except your classmates, and the instructor. However, the solutions should be written by you and you alone in your own words. Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

If you discuss the problems with anyone, please note their name at the top of your HW submission under a subtitle "Collaborator" or "Discussed with:".

Submit all 5 problems listed below.

1. Recall: Petersen Graph is the simple graph whose vertices are the 2-element subsets of the set $\{1, 2, 3, 4, 5\}$ and whose edges are pairs of disjoint 2-element subsets. Note that Petersen graph has 10 vertices and 15 edges.

a) Let u and v be non-adjacent vertices in the Petersen graph. Prove that u and v have exactly one neighbor.
Use this to show that Petersen graph has girth 5 (recall: girth is the length of a shortest cycle).

b) Is Petersen graph bipartite? Find the size of its largest independent set.

c) It can be shown (try it if you like) that: each edge in the Petersen graph belongs to exactly four 5-cycles. Use this fact to find a bipartite subgraph of the Petersen graph with maximum number of edges.

2. Let G be a connected graph with exactly $2k$ odd degree vertices, where $k \geq 1$. Prove that G can be decomposed using exactly k trails (walks that do not repeat any edges).

3. Prove that every tree with maximum degree $\Delta > 1$ has at least Δ leaves. Show this is best possible by constructing for each value of n and Δ, an n-vertex tree with exactly $\Delta = \Delta(G)$ leaves.

4. Let G be a connected graph and T, T' be two spanning trees of G.

a) For any $e \in E(T) - E(T')$, prove that there exists an $e' \in E(T') - E(T)$ such that $T - e + e'$ is a spanning tree of G.

b) For any $e \in E(T) - E(T')$, prove that there exists an $e' \in E(T') - E(T)$ such that $T' + e - e'$ is a spanning tree of G.

c) For any $e \in E(T) - E(T')$, prove that there exists an $e' \in E(T') - E(T)$ such that $T' + e - e'$ and $T - e + e'$ are both spanning trees of G.

[*Comment:* You should remember these statements, they are very useful.]

5. Prove any one of the following two statements:

(a) If no two edge weights of a connected graph G are equal, then G has a unique minimum spanning tree. [*Comment:* You can use Problem 4 even if you did not solve it.]

(b) Let e be the unique edge of minimum weight in a weighted connected graph G. Then e must be an edge in every minimum spanning tree of G.