MATH 553 : Homework #3

Due Thursday, 10/4, in class before the lecture starts.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with it, don't hesitate to ask. You can contact me during office hours, or through email.

All problems require explicit and detailed proofs/ arguments/ reasons. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

You are allowed to discuss the homework problems with no one except your classmates, and the instructor. However, the solutions should be written by you and you alone in your own words. Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

If you discuss the problems with anyone, please note their name at the top of your HW submission under a subtitle "Collaborator" or "Discussed with:".

Submit a total of 7 problems out of the 8 listed below. Problem 1 is compulsory.

1. (a) Which of the following relations is true for every G and H, where H is a subgraph of G? Assume both G and H have minimum degree at least 1.
(i) $\alpha(H) \leq \alpha(G)$; (ii) $\alpha'(H) \leq \alpha'(G)$; (iii) $\beta(H) \leq \beta(G)$; (iv) $\beta'(H) \leq \beta'(G)$.

(b) Let G be a graph with n vertices and m edges.
Show that $\alpha(G) \leq m/\delta(G)$ and $\alpha(G) \geq n/(\Delta(G) + 1)$.

2. For each $k \geq 2$, construct a k-regular graph with no perfect matching.

3. Prove that every bipartite graph G has a matching of size at least $|E(G)|/\Delta(G)$. Use this to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

4. Prove that $\alpha(G) \leq |V(G)| - |E(G)|/\Delta(G)$ for any non-trivial graph G. Conclude that $\alpha(G) \leq |V(G)|/2$ when G is also regular.

5. Two players play a game on a graph G, alternately choosing distinct vertices. Player 1 starts by choosing any vertex. Each choice of vertex in a move must be adjacent to the vertex chosen by the other player in the previous turn. That is, together the players are following a path. The last player to be able to make a move wins.

Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy.

6. Let $f(G)$ be the minimum number of vertices that are unsaturated by any matching in a graph G. Prove that $f(G) \le t$ if and only if $q(G-S) \le |S| + t$ for every subset S of $V(G)$.

7. Prove that a tree T has a perfect matching if and only if $q(T-v) = 1$ for every $v \in V(T)$.

8. Let G be a 3-regular graph with at most two bridges (cut-edges). Prove that G has a 1-factor.