<u>MATH 553 : Homework #5</u>

Due Thursday, 11/8, in class before the lecture starts.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with it, don't hesitate to ask. You can contact me during office hours, or through email.

All problems require explicit and detailed proofs/ arguments/ reasons. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

You are allowed to discuss the homework problems with no one except your classmates, and the instructor. However, the solutions should be written by you and you alone in your own words. Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

If you discuss the problems with anyone, please note their name at the top of your HW submission under a subtitle "Collaborator" or "Discussed with:".

Solve a total of 8 problems as required in Parts I and II.

PART I. Submit 4 out of the following 5 problems.

1. True or False? Give a short proof or a counterexample, as appropriate.
(a) A plane graph has a cutvertex if and only if its dual has a cutvertex.

(b) There is no simple bipartite planar graph with minimum degree at least 4.

2. (a) Let G be an n-vertex connected simple planar graph with girth k. Prove that G has at most $(n-2)k/(k-2)$ edges. Use this to prove that Petersen graph is nonplanar.
(b) Use Kuratowski's theorem to show that Petersen graph is nonplanar.

3. Find a sharp upper bound on the number of edges in a simple outerplanar graph on n vertices ($n > 2$) using each of the following methods:

a) By using induction on n.
b) By using Euler's formula.
c) By transforming G into a planar graph that is non-outerplanar: add a vertex in the unbounded face of G and add appropriate edges.

4. Determine $cr(K_{1,2,2,2})$ and use it to compute $cr(K_{2,2,2,2})$.

5. Use Kuratowski's theorem to prove that: G is outerplanar if and only if G contains no subdivision of K_4 or $K_{2,3}$.

PART II. Submit 4 out of the following 5 problems.

6. Let G be bipartite. Prove that $\chi(\overline{G}) = \omega(\overline{G})$ (without using theorems related to perfect graphs).

7. Let G be a graph with the property that every two odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$.

8. Let G be a graph on n vertices. Use induction on n to prove that: $\chi(G) + \chi(\overline{G}) \leq n + 1$.

9. Let H_1 and H_2 be two graphs with $V(H_1) = V(H_2)$. Let G be the union of H_1 and H_2, i.e., $V(G) = V(H_1) = V(H_2)$ and $E(G) = E(H_1) \cup E(H_2)$. Prove that $\chi(G) \leq \chi(H_1)\chi(H_2)$. Show that this bound is sharp by constructing an example with non-trivial H_1 and H_2.

10. (a) Prove that every graph G has a $\Delta(G)$-regular (loopless) supergraph.

(b) Let G be a graph such that $\Delta(G) = 2k$. Prove that $\chi'(G) \leq 3k$ without using Vizing-Gupta theorem.