MATH 553 : Homework #6

Due Thursday, 11/29, in class before the lecture starts.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with it, don't hesitate to ask. You can contact me during office hours, or through email.

All problems require explicit and detailed proofs/ arguments/ reasons. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

You are allowed to discuss the homework problems with no one except your classmates, and the instructor. However, the solutions should be written by you and you alone in your own words. Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

If you discuss the problems with anyone, please note their name at the top of your HW submission under a subtitle "Collaborator" or "Discussed with:".

Solve a total of 8 problems as required in Parts I and II.

PART I. Submit 4 out of the following 5 problems.

1. Prove the Art Gallery Theorem: If an art gallery is laid out as a simple (non-self-intersecting) polygon with n sides, then it is possible to place at most $n/3$ guards such that each point of the interior of the gallery is visible to at least one such guard.

[*Comment*: Model the art gallery as an outerplanar graph. Then transform this outerplanar graph so that its proper 3-coloring determines where to place the guards.]

2. Let G be a graph on six vertices that consists of a C_6 with vertices labeled clockwise as $v_1, \ldots, v_6$ and an additional edge between v_1 and v_4. Show that G is NOT 2-choosable (that is, the list chromatic number is strictly larger than 2).

3. Let G be a connected graph with lists of colors $L(v)$ assigned to each vertex v. If $|L(v)| \geq d(v)$ for all v and $|L(v)| > d(v)$ for at least one v, then prove that G has a proper L-list-coloring. ($d(v)$ denotes the degree of v.)

4. Each game of cards involves two teams of two partners each. Consider a club where four players can not play a game if any two have previously been a partner that night. One evening 14 members are present. First they play games until each member has played four times. Then, they are able to play six additional games. At the end of all these games, when they are about to stop playing, one new member arrives. Prove that at least one more game can be played now. [*H*int: Model the problem using a graph. Apply Mantel (or Turan's) theorem to it.]

5. Use the probabilistic method to prove that every graph G with m edges contains a bipartite subgraph with at least $m/2$ edges.

[*Comment*: Set up your experiment by "randomly" (?) deciding for each vertex which partite set A or B should it be placed in? What is the probability an edge has one endpoint in A and the other in B? What are the expected number of edges inbetween A and B?]

PART II. Submit 4 out of the following 5 problems.

1. Using induction on r (and not using the statement of Turán's theorem) prove that every n-vertex graph with no K_{r+1} has at most $(1/2)(1-1/r)n^2$ edges.

2. Prove that there exists an integer s such that every 2-coloring of the integers $\{1, \ldots, s\}$ yields monochromatic (but not necessarily distinct) $x, y, z \in \{1, \ldots, s\}$ solving $x + y = z$.
[*Comment*: Define a 2–coloring of edges of an appropriate K_{s+1} using the given 2-coloring of the integers $\{1, \ldots, s\}$. Choose your s in terms of $R(3, 3)$.]

3. Show the Graph Ramsey number $R(K_{1,m}, K_{1,n})$ equals $m + n$ unless m and n are both even in which case it equals $m + n - 1$.
[*Comment*: Think of pigeonhole principle applied to the number of potential neighbors and non-neighbors of a vertex.]

4. Let $p > (m - 1)(n - 1)$. Prove that: Given a 2-coloring (using colors red and blue) of $E(K_p)$ in which the red graph is transitively orientable, there must exist a red K_m or a blue K_n.
[*Comment*: Recall that a transitively orientable graph is called a comparability graph, which is a perfect graph. Use properties of perfect graphs in your proof.]

5. Let T be a tree with m edges (and so $m + 1$ vertices). Prove that $R(T, K_{1,m+1}) = 2m + 1$.
[*Comment*: Use the statement (can you prove it by induction on k?): Let T be a tree with k edges. If G is graph with $\delta(G) \geq k$ then G must contain T as a subgraph. Note that this is Corollary 1.5.4 in your textbook.]