MATH 400: Homework #1

Only the 'Submission Problems' listed below are due Wednesday, 9/2, before 11:59pm, via a PDF file uploaded to the Homework#1 under Assignments in the Blackboard course page.

You are allowed to discuss the homework problems with no one except your classmates, the TA, and the instructor. However, the solutions should be written by you and you alone in your own words. **If you discussed HW problems with a classmate or TA, you have to write their name at the top of the HW submission as a collaborator.** Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

Re-read the "Homework Assignment" and " 'Why and How' of Homework" sections of the course information sheet for some important advice on the HWs for this course.

All problems require explicit and detailed explanations. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with any HW problem, don't hesitate to ask me. You are encouraged to ask questions during the *Live Class on Blackboard*, through the *Blackboard Discussion Forums*, during the *Google Meet Office Hours*, during the *TA office hours*, or through *Email to me*.

PART I: Practice Problems

1. Read and think about all the exercises in Sections 1.2 and 1.3 of the textbook.
2. Pay closer attention to and attempt at least the following exercises.
   Section 1.2: #1, #3, #5, #6, #7, #9, #10, #11, #13.
   Section 1.3: #1, #3, #4, #5, #8, #9, #11.

PART II: Submission Problems

3. **Submit written solutions to any 5 out of the following 6 exercises:**

   Section 1.2: #1, #11, #13.

   Section 1.3: #3, #5, #9.

PART III: Readings, Comments, etc.

4. We saw a simple proof that $\sqrt{2}$ is irrational. What about π, e, π^2, e^r for any $r \in \mathbb{Q}^+$? All of these numbers can be shown to be irrational using arguments based just on calculus. The best source for this (and many other elegant proofs) is *Proofs from THE BOOK* by M. Aigner and G.M. Ziegler (Chapter 8 in the 6th edition). Look it up!

5. A real number is called an *algebraic number* if it is a root of a nonconstant polynomial with rational coefficients. Let $\mathbb{A}$ be the set of all such real numbers. Then we can immediately deduce that

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{A} \subseteq \mathbb{R}$$

   Recall the definition of $\mathbb{Q}$ as the roots of all linear equations with integer coefficients. $\mathbb{A}$ contains all the roots of all polynomials with rational coefficients! Its natural to ask whether $\mathbb{A} = \mathbb{R}$?

   Let $\mathbb{I} = \mathbb{R} \setminus \mathbb{Q}$ be the set of irrational numbers. Let $\mathbb{T} = \mathbb{R} \setminus \mathbb{A}$ be the set of transcendental numbers. Another way of asking the earlier question is whether $\mathbb{T}$ is nonempty. The answer is yes but, unlike $\mathbb{I}$, its quite hard to show (it wasn't proved till 1844). Even proving that a specific number like π or e is transcendental is not easy (shown only in 1882 and 1873 respectively). Its still not known whether $e\pi$ and e^π are transcendental (its known that at least one of them is!). In fact, most sums, products, powers of e and π such as $\pi + e$, $\pi - e$, πe, π/e, π^π, π^e, e^e are still not known to be transcendental or even irrational. It is known though that (in a very precise mathematical sense) almost all real numbers are transcendental. But finding them is not easy. For mathematicians, its often hard to find hay in a haystack :-)

6. Let me end with asking you a very natural question. Consider all eight possible questions of the form: whether there exist $x \in \mathbb{I}$ or $\mathbb{Q}$, and $y \in \mathbb{I}$ or $\mathbb{Q}$, such that x^y belongs to $\mathbb{I}$ or $\mathbb{Q}$?

   For example, we know the answer is yes for the question: Does there exist $x \in \mathbb{Q}$, and $y \in \mathbb{Q}$, such that $x^y \in \mathbb{I}$? Right?

   What about: Does there exist $x \in \mathbb{Q}$, and $y \in \mathbb{Q}$, such that $x^y \in \mathbb{Q}$? That should be easy :-)

   The following are also easy: Does there exist $x \in \mathbb{I}$, and $y \in \mathbb{Q}$, such that $x^y \in \mathbb{I}$? Does there exist $x \in \mathbb{I}$, and $y \in \mathbb{Q}$, such that $x^y \in \mathbb{Q}$?

   My question for you is: Does there exist $x \in \mathbb{I}$, and $y \in \mathbb{I}$, such that $x^y \in \mathbb{Q}$?