MATH 400: Homework #2

Only the 'Submission Problems' listed below are due Wednesday, 9/9, before 11:59pm, via a PDF file uploaded to the Homework#2 under Assignments in the Blackboard course page.

You are allowed to discuss the homework problems with no one except your classmates, the TA, and the instructor. However, the solutions should be written by you and you alone in your own words. **If you discussed HW problems with a classmate or TA, you have to write their name at the top of the HW submission as a collaborator.** Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

Re-read the "Homework Assignment" and " 'Why and How' of Homework" sections of the course information sheet for some important advice on the HWs for this course.

All problems require explicit and detailed explanations. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with any HW problem, don't hesitate to ask me. You are encouraged to ask questions during the *Live Class on Blackboard*, through the *Blackboard Discussion Forums*, during the *Google Meet Office Hours*, during the *TA office hours*, or through *Email to me*.

PART I: Practice Problems

1. Pay close attention to and attempt at least the following exercises from the textbook.
   Section 1.4: #1, #3, #5, #7, #8.
   Section 1.5: #1, #3, #4, #5, #7, #9, #11 (Note this statement and use it if needed).
   Section 1.6: #1, #4, #9, #10ab.

PART II: Submission Problems

2. **Submit written solutions to 4 exercises as listed below:**
   Section 1.4: Submit one of #5 or #7.
   Section 1.5: Submit any two of #3, or #7, or #9.
   Section 1.6: #10ab.

PART III: Readings, Comments, etc.

3. Is it possible to have an unprovable mathematical statement? Please note that I used the word 'unprovable', not 'disprovable' or 'provable'. To understand what this means, we have ask ourselves how do we prove statements in mathematics. Every theorem that we prove is ultimately dependent on fundamental assumptions we make about the mathematical structures that we are studying. These fundamental assumptions, these 'self-evident truths' are called 'axioms'.

You might have studied Euclid's five axioms for geometry in high school which give us a foundation for building Euclidean plane geometry. The fifth axiom in this list is called the parallel postulate - *For every straight line, there is exactly one line passing through a given point that is parallel to it*. In the nineteenth century, it was proven that this parallel postulate is *independent* of (or *unprovable* under) other 4 axioms. That is, (i) the first four axioms do not disprove the parallel postulate, and (ii) the first four axioms do not disprove the negation of the parallel postulate. For example, Hyperbolic geometry replaces the axiom of parallel lines in Euclidean geometry by allowing multiple distinct parallel lines to a given line.

4. Cantor asked a question about cardinalities that was later called:
   The Continuum Hypothesis (CH): There is no set S with $|\mathbb{N}| < |S| < |\mathbb{R}|$.
   Cantor strongly believed that CH was true, but was never able to prove it. In mid-twentieth century it was shown that CH is unprovable. Unprovable under what system of axioms? For most of modern mathematics, the commonly used system of axioms is called ZFC (Zermelo-Fraenkel with axiom of Choice). Gödel (1940) showed that CH can not be disproved from ZFC. Cohen (1964) showed that CH can not be proven from ZFC. Such a basic question about sizes of infinite sets has no definitive answer in a sense!

5. ZFC is the modern axiomatic system of mathematics that was built to overcome the various paradoxes inherent in naive definitions of sets, the building block of mathematics. Russell's paradox is the contradiction that arises when we define a set R to be the set of all sets that are not members of themselves. Does R belong to R or not? A version of this paradox is sometimes described as 'Who shaves the barber? Where a barber is the person who shaves those who do not shave themselves.' Also, the liar paradox: "This sentence is false.", is it or not? ZFC was an important step towards showing that foundations of mathematics were secure and all of practical mathematics can be consistently and precisely developed from some small collection of fundamental axioms.

6. Kurt Gödel was one the greatest mathematicians/ logicians of the twentieth century. He showed that a complete axiomatization of mathematics was an impossible task in a certain strong sense. Gödel's first incompleteness theorem (1931) states (informally): Any consistent system of axioms that is rich enough to do arithmetic, will always contain unprovable statements. Note that this theorem applies to any such system of axioms!! The essential part of the proof is called the Diagonal Lemma which is similar in flavor to Cantor's diagonalization argument. If you are a computer science student or interested in foundations of computation, look up the 'Halting Problem' and the proof of its 'undecidability'.

7. Add the book *Gödel, Escher, Bach: An Eternal Golden Braid* by Douglas Hofstadter, to your reading list if you haven't read it yet.

8. A couple of fun problems which have connections toour studies this week:

   (a) Suppose a rabbit is moving along a straight line to the lattice points (points with both coordinates being integers) of the plane by making identical jumps every minute (but we don't know where it is and what kind of jump it is making). We can place a trap every hour at an arbitrary lattice point of the plane of our choice. Show that we will ultimately capture the rabbit.

   (b) Let $A \subset [0, 1]$ be a set. Two players Alice and Bob play a game as: they alternatively select digits (numbers 0 to 9) $x_0, x_1, x_2, \ldots$ and $y_0, y_1, y_2, \ldots$, respectively. Alice wins if the number $0.x_1y_1x_2y_2\ldots$ is in A, otherwise Bob wins. Show that Bob has a winning strategy if A is countable.