# MATH 400: Homework #4

**Only the 'Submission Problems' listed below are due Wednesday, 9/23, before 11:59pm, via a PDF file uploaded to the Homework#3 under Assignments in the Blackboard course page.**

You are allowed to discuss the homework problems with no one except your classmates, the TA, and the instructor. However, the solutions should be written by you and you alone in your own words. **If you discussed HW problems with a classmate or TA, you have to write their name at the top of the HW submission as a collaborator.** Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

Re-read the "Homework Assignment" and " 'Why and How' of Homework" sections of the course information sheet for some important advice on the HWs for this course.

All problems require explicit and detailed explanations. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with any HW problem, don't hesitate to ask me. You are encouraged to ask questions during the *Live Class on Blackboard*, through the *Blackboard Discussion Forums*, during the *Google Meet Office Hours*, during the *TA office hours*, or through *Email to me*.

## PART I: Practice Problems

1. Pay close attention to and attempt at least the following exercises from the textbook.
   Section 2.4: #1, #2, #3, #5, #6, #7 (read the definition of limit superior even if you don't try to solve the problem), #8.
   Section 2.5: #1, #2, #3, #5, #6, #7.

2. We say a real number $K$ is a *cluster point* of a sequence $(a_n)$ if $K$ is the limit of a convergent subsequence of $(a_n)$. Find all the cluster points of the sequence $(\sin(n\pi/2))$.

## PART II: Submission Problems

3. **Submit written solutions to any 5 out of the 6 problems as listed below. Note that you are allowed to use the squeeze theorem (exercise 2.3.3) and exercise 2.3.1 from previous HWs.**

   - Section 2.4: #3a, #6, #8c.
   - Section 2.5: #3, #6&7 (count as one problem; you can use any method/idea).
   - [Special Problem; This problem is compulsory.]
     Assume $\int_0^1 x^n(1-x)^n dx$ exists for every integer $n \geq 1$ (you know this from Calculus). Let $a_n = \int_0^1 x^n(1-x)^n dx$. Show that $\lim_{n\to\infty} a_n = 0$, without evaluating the integral. (First read and understand the example in item 4 in PART III.)

**PART III: Readings, Comments, etc.**

4. Definite integrals are essentially real numbers and sequences of definite integrals can be analyzed (as functions of some integer parameter $n$) using the techniques of sequence convergence and divergence.

   To illustrate this, let us consider the integral $\int_0^1 (x^2+2)^n dx$. We know from calculus that this integral exists for each fixed positive integer $n$. We are interested in understanding $\lim_{n\to\infty} \int_0^1 (x^2+2)^n dx$. We can, of course, first evaluate the integral but that would just yield an expression in $n$ that is difficult to estimate and analyze using the methods of sequence convergence. Instead we estimate the integrand and use the bound to estimate the integral, and consequently the sequence $a_n = \int_0^1 (x^2+2)^n dx$.

   We know $x^2+2 \geq 2$ for all $x$ (why?). Therefore, $(x^2+2)^n \geq 2^n$ for all $x$ and all $n \geq 1$. Thus, $\int_0^1 (x^2+2)^n dx \geq \int_0^1 2^n dx = 2^n(1-0) = 2^n$. (why is the inequality true?)
   Since $\lim 2^n = \infty$, the definite integral must also tend to $\infty$. Let us verify the definition of 'divergence to $\infty$' as we gave in a lecture.
   For any $M > 0$, $\int_0^1 (x^2+2)^n dx \geq 2^n \geq M$ for every $n > \log_2 M$.
   Thus, $\lim_{n\to\infty} \int_0^1 (x^2+2)^n dx = \infty$.

5. Try this interesting problem of finding the limit of an integral.
   Let $a_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$. Determine $\lim_{n\to\infty} a_n$.
   First sketch the curve and look at the area under the curve to guess what the long-term behavior of the integral must be. Then prove that your guess is in fact true by a proof of convergence.