# MATH 400: Homework #5

**Only the 'Submission Problems' listed below are due Wednesday, 9/30, before 11:59pm, via a PDF file uploaded to the Homework#5 under Assignments in the Blackboard course page.**

You are allowed to discuss the homework problems with no one except your classmates, the TA, and the instructor. However, the solutions should be written by you and you alone in your own words. **If you discussed HW problems with a classmate or TA, you have to write their name at the top of the HW submission as a collaborator.** Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

Re-read the "Homework Assignment" and " 'Why and How' of Homework" sections of the course information sheet for some important advice on the HWs for this course.

All problems require explicit and detailed explanations. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with any HW problem, don't hesitate to ask me. You are encouraged to ask questions during the *Live Class on Blackboard*, through the *Blackboard Discussion Forums*, during the *Google Meet Office Hours*, during the *TA office hours*, or through *Email to me*.

## PART I: Practice Problems

1. Pay close attention to and attempt at least the following exercises from the textbook.
   Section 2.6: #1, #2, #3, #5.
   Section 2.7: #1, #2, #3, #4, #5, #7, #9, #11.
2. Prove that the sequence $(n^{1/n})$ converges to 1. (I asked this as an optional problem in HW#3 but now you are ready to solve it. Use the binomial theorem as we did in HW#4.)

## PART II: Submission Problems

3. **Submit written solutions to 5 out of the 6 problems as listed below.**
   - Section 2.6: #3a.
   - Section 2.7: #1ac, #3b, #9 (This problem is compulsory).
   - [Special Problem A: This problem is compulsory.]
     Consider the sequence $(a_n)$ defined recursively as $a_{n+1} = \frac{1}{a_n+1}$ for $n \geq 1$ and $a_1 = 1$. (Do the terms of this sequence remind you of some well-known sequence?)
     (a) Show that $a_n \geq 1/2$ for all $n$.
     (b) Show that $|a_n - a_{n+1}| \leq 1/2^n$ for all $n$.
     (c) Show that $(a_n)$ is a Cauchy sequence.
     (d) Determine the limit of $(a_n)$, if it exists.

- [Special Problem B]
  For any fixed real number $x$, we define a series as $S(x) = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{x^{2k-1}}{2k-1}$.
  Prove that the series $S(x)$ converges iff $-1 < x \leq 1$. (Note: you can use the ratio test here.)

**PART III: Readings, Comments, etc.**

4. Our course started with a discussion about shortcomings of $\mathbb{Q}$ which we overcame by using $\mathbb{R}$ instead of $\mathbb{Q}$, as $\mathbb{R}$ has no "holes". We formalized this notion of "no holes" using the language of Axiom of Completeness (AoC). Using AoC we were able to prove Montotone Convergence Test (MCT), which in turn helped us prove the Bolzano-Weierstrass theorem (BW) and finally lead to the Cauchy Criterion (CC). That is,
   $$\text{AoC} \implies \text{MCT} \implies \text{B-W} \implies \text{CC}.$$
   It turns out that AoC and B-W are equivalent (try it!). And so are AoC and MCT (try it!). So we could have replaced AoC with either one of MCT or B-W at the beginning, and still have developed the same theory of real numbers and their sequences.

   What about CC? Unfortunately CC by itself is not strong enough to prove AoC. We also need the Archimedean Principle (AP). With AP we have the following:
   $$\text{AoC} \iff \text{MCT} \iff \text{B-W} \iff (\text{CC} + \text{AP}).$$

5. Although mathematicians often talk about a theorem implies another theorem , or two theorems being equivalent, we are quite informal in our understanding of relative "strength" of a theorem compared to another. To truly understand how a particular theorem is proved using the conclusion of another theorem, we need to carefully delineate the underlying axioms that lead us to each of the theorems. **Reverse Mathematics** is the field within mathematical logic that studies which axioms are essential for the proof of a theorem.

   Starting from a base axiom system that strong enough to develop the definitions underlying a theory but not strong enough to prove the theory itself, we look for a minimal set of axioms to add to the base axioms so that theorem can be proved. In a sense, this shows the equivalence of the extended axiom system and the theorem (one implication is the usual proof of a theorem using axioms, while the other implication is the identification of axioms necessary for the proof of the theorem). If this intrigues you, then look up the book: *J. Stillwell (2018), Reverse Mathematics, proofs from the inside out*.

6. Some Solved Problems:

   (i) In the early days of Calculus, a seemingly simple problem remained unsolved despite many efforts by the leading mathematicians of the day. What is $\sum_{n=1}^{\infty}\frac{1}{n^2}$? In 1734, a young Euler proved that it equals $\frac{\pi^2}{6}$ (!!). There have been many proofs given of this fact. A simple one is based on two different evaluations of the double integral $\int_0^1\int_0^1 \frac{1}{1-xy}dxdy$. Try it! OR look up this and three other proofs in a book that I recommended earlier also: *Proofs from THE BOOK* by M. Aigner and G.M. Ziegler (Chapter 9 in the 6th edition).

   (ii) What does $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ equal? Integrating the each term of the geometric series $\sum_{n=1}^{\infty}(-1)^{n+1}x^n = \frac{1}{1+x}$, and then evaluating the resultant identity at $x = 1$ (why is that justified?) gives us $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$.

(iii) Every positive rational number can be written as a sum of distinct numbers of the form $1/n$. Try it for some of your favorite fractions. Why is it always true?

7. Some Unsolved Problems:

(i) For any natural number $a_0$, define a sequence as follows

$$a_{n+1} = \begin{cases} a_n/2 & \text{if } a_n \text{ is even} \\ (3a_n + 1) & \text{if } a_n \text{ is odd} \end{cases}$$

Is such a sequence always bounded, regardless of what $a_0$ is? Will this sequence always eventually reach 1, regardless of what $a_0$ is? (Warning: Do not spend too much time working on this! This easy to describe and easy to compute problem has resisted all efforts for a solution for the past 80+ years. Ask me if you are curious to learn more about it.)

(ii) Does $\sum_{n=1}^{\infty} \frac{1}{n^3 \sin^2(n)}$ converge?

(iii) Does $\sum_{n=1}^{\infty} \frac{(-1)^n n}{p_n}$ converge, where $p_n$ is the $n$th prime number?

(iv) Is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ irrational?

(v) Is $\sum_{n=0}^{\infty} \frac{1}{n^3}$ transcendental?

Nobody has been able to resolve any of these problems so far. How about you?