Welcome to

MATH 400 Real Analysis.

Video #1

We are concerned about

→ What is a real number?

→ How do we find the limit of a sequence of numbers?

→ Can we add infinitely many real numbers together and still get a finite number?

→ Can we rearrange the elements of an infinite sum, and still have the same sum?

→ What is a function?
~~What is~~ does it mean for a function to be continuous? differentiable? integrable? bounded?

Limitations of Calculus

① Division by zero

Cancellation law $\quad ac = bc \Rightarrow a = b$ does not work when $c = 0$

But often times such division by zero can be hidden.

In calculus, we have to be careful about defining functions and limits when a denominator could become zero (or "close" to zero).

(2) Divergent Series

Recall $S = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

then using the trick $2S = 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

$= 2 + S$

& hence $S = 2$

Apply the same trick to the series

$S = 1 + 2 + 4 + 8 + 16 + \cdots$

$\Rightarrow 2S = 2 + 4 + 8 + 16 + \cdots = S - 1$

$\Rightarrow S = -1$

Another example: $S = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$

$S = (1-1) + (1-1) + \cdots = 0 + 0 + 0 + \cdots = 0$

$S = 1 - (1 - 1 + 1 - 1 + 1 - 1 + \cdots) = 1 - S \Rightarrow S = 1/2$

$\frac{1}{2} = 0$??

③ Divergent sequences

Let r be a fixed real number.

Let $L = \lim_{n\to\infty} r^n$

Changing variables $n = m+1$,

$$L = \lim_{m+1\to\infty} r^{m+1} = \lim_{m+1\to\infty} r\, r^m = r \lim_{m+1\to\infty} r^m$$

$$= r \lim_{m\to\infty} r^m = r L$$

$\Rightarrow$ $rL = L$

$\Rightarrow$ either $r = 1$ or $L = 0$

$\Rightarrow$ $\lim_{n\to\infty} r^n = 0$ if $r \neq 1$??

4 Limiting values of functions

Recall $\sin(y+\pi) = -\sin(y)$

So, $\lim\limits_{x\to\infty} \sin(x) = \lim\limits_{y+\pi\to\infty} \sin(y+\pi) = \lim\limits_{y\to\infty} (-\sin(y)) = -\lim\limits_{y\to\infty} \sin(y)$

$\Rightarrow \lim\limits_{x\to\infty} \sin(x) = -\lim\limits_{x\to\infty} \sin(x) \quad \Rightarrow \lim\limits_{x\to\infty} \sin(x) = 0$

Recall $\sin(z+\frac{\pi}{2}) = \cos(z)$

so, $\lim\limits_{x\to\infty} \sin(x) = \ ?\ ?\ ?\ ? = \lim\limits_{z\to\infty} \cos(z)$

$\Rightarrow \lim\limits_{x\to\infty} \cos(x) = 0$

$\therefore \lim\limits_{x\to\infty} (\sin^2(x) + \cos^2(x)) = 0^2 + 0^2 = 0$

BUT!! $\sin^2 x + \cos^2 x = 1 \ \forall x \Rightarrow \lim\limits_{x\to\infty} \sin^2 x + \cos^2 x = 1$

$0 = 1$??

⑤ Interchanging Sums

Take a matrix of numbers, e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ & compute the sums of all rows and all columns & then total those.

$$\begin{array}{cc} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} & \begin{array}{c} 6 \\ 15 \\ 24 \end{array} \\ \begin{array}{ccc} 12 & 15 & 18 \end{array} & 45 \end{array}$$

$$\sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}$$

But what about $\displaystyle\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij}$?

$$\begin{array}{cc} \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots \\ -1 & 1 & 0 & 0 & \cdots \\ 0 & -1 & 1 & 0 & \cdots \\ 0 & 0 & -1 & 1 & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} & \begin{array}{c} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \end{array} \\ \begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \end{array} & \end{array}$$

$$1 = 0 \;??$$

6 Interchanging integrals

Recall computing volume under a surface $z = f(x,y)$

By slicing parallel to x-axis for each fixed value of y, we compute the area $\int f(x,y)\,dx$ & then integrate that area in y to get: Volume $= \int\int f(x,y)\,dx\,dy$

By slicing parallel to y-axis: Volume $= \int\int f(x,y)\,dy\,dx$

Can we always interchange integrals?

$$\int_0^1 (e^{-xy} - xy e^{-xy})\,dy = y e^{-xy}\Big|_{y=0}^{y=1} = e^{-x}$$

gives us

and
$$\int_0^\infty (e^{-xy} - xy e^{-xy})\,dx = x e^{-xy}\Big|_{x=0}^{x=\infty} = 0$$

① $$\boxed{\int_0^\infty \int_0^1 (e^{-xy} - xy e^{-xy})\,dy\,dx} = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = 1$$

② $$\boxed{\int_0^1 \int_0^\infty (e^{-xy} - xy e^{-xy})\,dx\,dy} = \int_0^1 0\,dx = 0$$

$1 = 0$??

7 Interchanging Limits

$$\lim_{x\to 0}\lim_{y\to 0}\frac{x^2}{x^2+y^2} = \lim_{y\to 0}\lim_{x\to 0}\frac{x^2}{x^2+y^2}$$

Note $\to$ $\displaystyle\lim_{y\to 0}\frac{x^2}{x^2+y^2} = \frac{x^2}{x^2+0^2} = 1 \Rightarrow \lim_{x\to 0}\lim_{y\to 0}\frac{x^2}{x^2+y^2} = 1$

$$\lim_{x\to 0}\frac{x^2}{x^2+y^2} = \frac{0^2}{0^2+y^2} = 0 \Rightarrow \lim_{y\to 0}\lim_{x\to 0}\frac{x^2}{x^2+y^2} = 0$$

$0 = 1$??

⑧ Interchanging limits and integrals

For any $y \in \mathbb{R}$, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2}\,dx = \arctan(x-y)\Big|_{x=-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right)$

$= \pi$

Taking limits as $y \to \infty$, we get

$$\int_{-\infty}^{\infty} \lim_{y\to\infty} \frac{1}{1+(x-y)^2}\,dx = \lim_{y\to\infty} \int_{-\infty}^{\infty} \frac{1}{1+(x-y)^2}\,dx = \pi$$

But for every x, $\displaystyle\lim_{y\to\infty} \frac{1}{1+(x-y)^2} = 0$

$$\Rightarrow \int_{-\infty}^{\infty} \lim_{y\to\infty} \frac{1}{1+(x-y)^2}\,dx = \int_{-\infty}^{\infty} 0\,dx = 0$$

$\pi = 0$??

9 L'Hopital's Rule

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)} \quad \text{if } f(x) \;\&\; g(x) \to 0 \text{ as } x \to x_0$$

e.g.

$$\lim_{x\to 0}\frac{x^2\sin(x^{-4})}{x} = \lim_{x\to 0} x\sin(x^{-4}) = 0$$

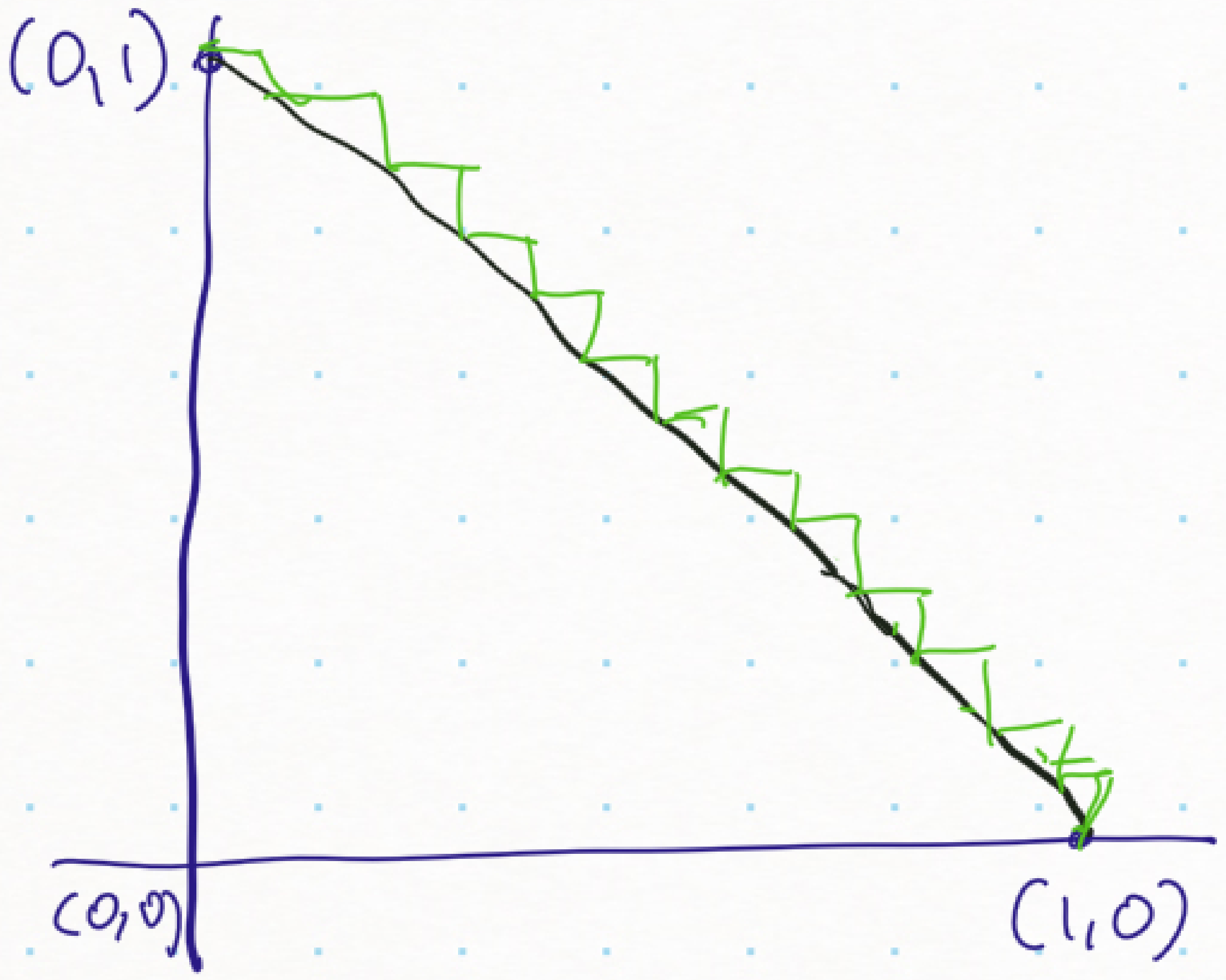
$$= \lim_{x\to 0}\frac{2x\sin(x^{-4}) - 4x^{-3}\cos(x^{-4})}{1}$$

$$= \underbrace{\lim_{x\to 0} 2x\sin(x^{-4}) - \lim_{x\to 0}\frac{4\cos(x^{-4})}{x^{3}}}_{\text{is divergent}}$$

$\lim_{x\to 0} 2x\sin(x^{-4}) \downarrow 0$ by squeeze test (try it!)

$\lim_{x\to 0}\frac{4\cos(x^{-4})}{x^{3}} \downarrow$ divergent

⑩ Limits and Lengths



What is the length of the hypotenuse?

Pick a large N and approximate the hypotenuse by a "staircase" of N horizontal edges & N vertical edges of the same length $(= 1/N)$

$\therefore$ Total length of the staircase $= 2\left(\frac{1}{N}\right)(N)$

As $N \to \infty$, "staircase" becomes closer & closer to the hypotenuse, so length of the hypotenuse

$$= \lim_{N\to\infty} \left(2 \frac{1}{N} N\right) = \lim_{N\to\infty} 2 = 2$$

By Pythagoras, $\sqrt{2}$!

MATH 400 Real Analysis

Video #2

"Natural numbers are the work of God. All the rest is the work of mankind"

— Kronecker (1823-1891)

$\mathbb{N} = \{1, 2, 3, \ldots\ldots\}$ sometimes w. 0

$\mathbb{Z}$ as solutions to $x + k = 0$, $k \in \mathbb{N}$

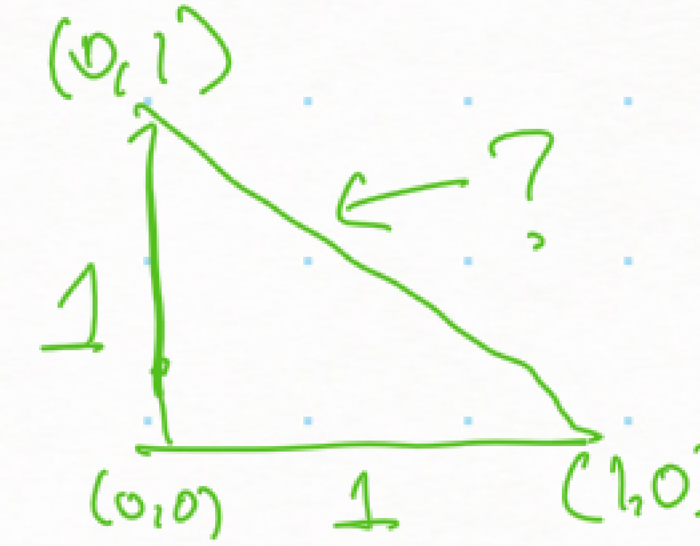
$\mathbb{Q}$ as solutions to $k_1 x + k_2 = 0$, $k_1, k_2 \in \mathbb{Z}$

$\mathbb{R}$ as ??

$\mathbb{C} = \{r_1 + i r_2 \mid r_1, r_2 \in \mathbb{R}\}$ arose out of attempts to define solutions to equations such as $x^2 + 1 = 0$, etc.

[But there is more than that in $\mathbb{C}$]

Is $\mathbb{R}$ the same as $\mathbb{Q}$?



<u>Thm</u> There is no solution in $\mathbb{Q}$ for the equation $x^2=2$

There is no rational number whose square is 2.

<u>Proof</u> For contradiction, assume $\exists\, p, q \in \mathbb{Z}$ s.t. $\left(\frac{p}{q}\right)^2 = 2$

we may assume that p & q have no common factor & write the fraction in lowest terms.

$p^2 = 2q^2 \Rightarrow p^2$ is even $\Rightarrow p$ is even

$\Rightarrow 2 \mid p$

$\Rightarrow p = 2r$

$p = 2k+1$

$p^2 = (2k+1)^2$

$= 4k^2+1+4k$

$= 2(2k^2+2k)+1$

$\Rightarrow (2r)^2 = 2q^2$

$\Rightarrow 4r^2 = 2q^2$

$\Rightarrow q^2 = 2r^2 \Rightarrow q^2$ is even $\Rightarrow q$ is even

$\Rightarrow 2 \mid q$

$\rightarrow p$ & q have a common factor

$\times$

$\mathbb{N}$ allows addition but no subtraction

$\mathbb{Z}$ allows addition & subtraction and multiplication but no division

$\mathbb{Q}$ allows addition & subt. and mult. & division.

? ✓ closed under the op.
✓ additive id. & ✓ additive inverses
✓ multiplicative id. & ✓ mult. inverses
✓ commutative $xy = yx$ $x+y = y+x$
✓ associative $(xy)z = x(yz)$
✓ distributive $a(b+c) = ab+ac$

} Field

$\mathbb{Q}$ also has a natural order on it : $r, s \in \mathbb{Q} \Rightarrow$ exactly one of following is true
$r < s$,
$r = s$
$r > s$.

Plus $r < s$, $s < t \Rightarrow r < t$
Transitive property.

$\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ all have a natural order

$\mathbb{N}, \mathbb{Z}$ are not fields but $\mathbb{Q}$ is a field

$\mathbb{N}, \mathbb{Z}$ have intervals of gaps, but $\mathbb{Q}$ is dense

Given any two $r < s \in \mathbb{Q}$, there is a $t \in \mathbb{Q}$ s.t. $r < t < s$

$$t = \frac{r+s}{2}$$

So what does $\mathbb{Q}$ lack?

It has lots of holes like $\sqrt{2}, \sqrt{3}, \ldots$

Although note that we can approximate these irrational numbers quite well using rationals

e.g. $(1.41)^2 = 1.9881$

$(1.414)^2 = 1.999396$ closer & closer to "$\sqrt{2}$"

$\ldots$

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Video # 3

Review notation & Terminology from Section 1.2

→ Induction example 1.2.7

→ Sets examples 1.2.1 & 1.2.2

$A \cup B$, $A \cap B$, A^c

$A \subseteq B$

$\bigcup_{i=1}^{\infty} A_i$ $\bigcap_{i=1}^{\infty} A_i$

De Morgan's Laws

① $(A \cap B)^c = A^c \cup B^c$

② $(A \cup B)^c = A^c \cap B^c$

Functions

(Dirichlet 1830s)

Domain coDomain

$$f : A \longrightarrow B$$

$$\underset{\in A}{x} \longmapsto \underset{\in B}{f(x)}$$

$$\text{Range}(f) = \{ y \in B \mid y = f(x) \text{ for some } x \in A \}$$

Beyond the notion of "f is a formula/expression"

example (Dirichlet 1829)

$$g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

$$g : \mathbb{R} \longrightarrow \mathbb{R}$$

$$\text{Range}(g) = \{0, 1\}$$

A special function

Absolute value function

$|\cdot| : \mathbb{R} \longrightarrow \mathbb{R}^+ \cup \{0\}$ } non-negative reals

positive reals $= \{x \in \mathbb{R} \mid x > 0\}$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

distance of x from 0

e.g.

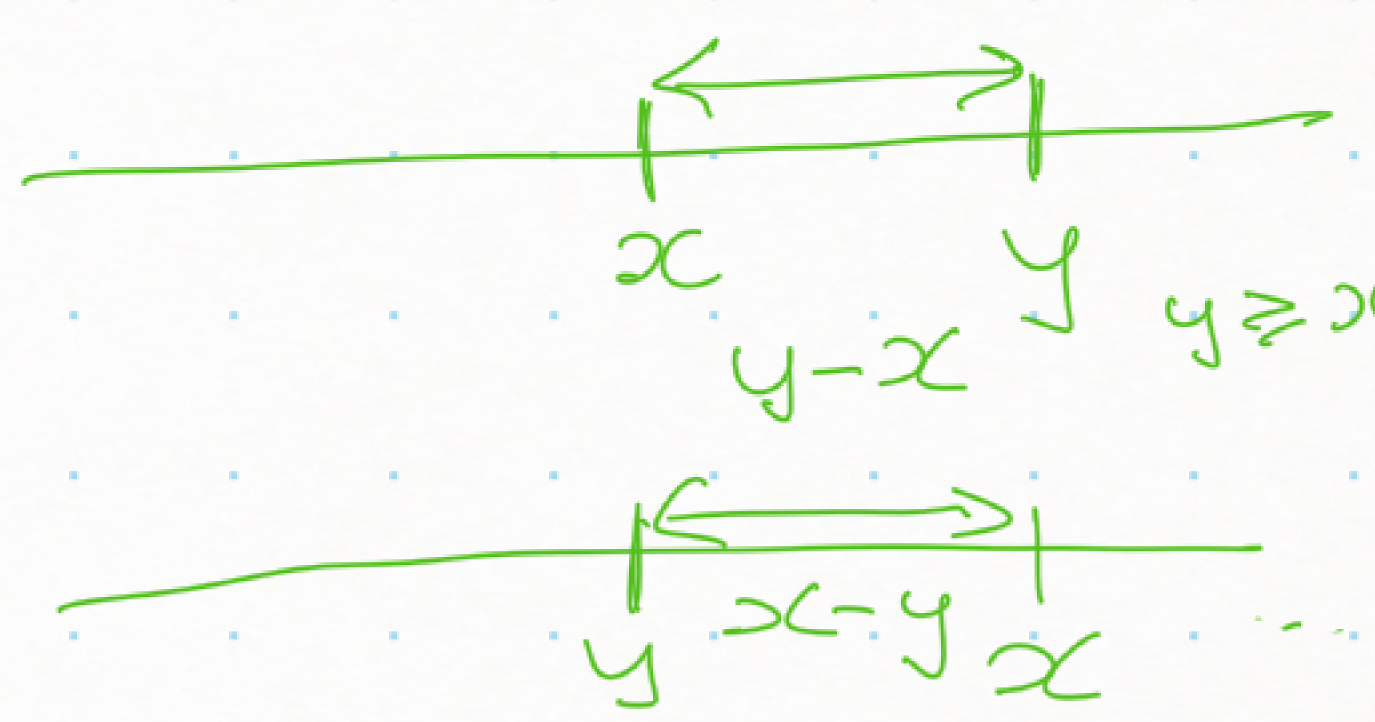
Properties

① $|ab| = |a||b|$

② Triangle Inequality $|a+b| \leq |a| + |b|$

distance between x & y

$\boxed{d(x,y) = |x-y|}$



Corollary Let $a, b, c \in \mathbb{R}$

$|a-b| \leq |a-c| + |c-b|$

Pf: $|a-b| = |(a-c) + (c-b)|$

$\leq |a-c| + |c-b|$

An example of a "Real Analysis-type" proof

$1 = 0.999\ldots$

$\epsilon = 0.00001$

$|1 - 0.9999\ldots|$
$= 0.0000000\ldots$
< 0.0001

Thm Let $a, b \in \mathbb{R}$.

$a = b$ if and only if $|a-b| < \epsilon \quad \forall \epsilon > 0$

Two real numbers a & b are equal iff for every real number $\epsilon > 0$ we have $|a-b| < \epsilon$

Proof $\boxed{\Rightarrow}$ Assume $a = b$

$|a-b| = 0 < \epsilon$ for every $\epsilon > 0$.

$\boxed{\Leftarrow}$ $|a-b| < \epsilon \ \forall \epsilon > 0 \Rightarrow a = b$

Contrapositive $a \neq b \Rightarrow \exists \epsilon_0 > 0$ s.t. $\boxed{|a-b| \geq \epsilon_0}$

Let $\epsilon_0 = |a-b| > 0$ (which is positive since $a \neq b$)

$\Rightarrow |a-b| \geq \epsilon_0$ ✓

Scratch

define $\epsilon_0 > 0$ using a, b & compare it to $|a-b|$

$|a-b| \neq 0$

$|a-b| > 0$

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Video #4

Recall, we think of $\mathbb{R}$ as an extension of $\mathbb{Q}$ in which there are no "holes" (like $\sqrt{2}$); every length along the number line corresponds to a real number.

<u>More formally</u>

$\mathbb{R}$ is a complete ordered field which contains $\mathbb{Q}$

Field addition, multiplication, inverses under both, 0 & 1, commutative, associative, distributive.

Ordered For $r, s, t \in \mathbb{R}$

$\rightarrow r < s$ or $r = s$ or $r > s$

$\rightarrow r \leq s$ & $s \leq r \Rightarrow r = s$

$\rightarrow r \leq s$ & $s \leq t \Rightarrow r \leq t$

$\rightarrow r \leq s \Rightarrow r + t \leq s + t$

$\rightarrow r \geq 0$ & $s \geq 0 \Rightarrow rs \geq 0$

There is a notion of positive numbers

Complete Satisfies "Axiom" of completeness

Every non empty set of real numbers that is bounded above has a least upper bound.

upper bound

Defn $A \subseteq \mathbb{R}$ is <u>bounded above</u> if $\exists\, b \in \mathbb{R}$ s.t. $a \leq b \ \forall a \in A$

$A \subseteq \mathbb{R}$ is <u>bounded below</u> if $\exists\, l \in \mathbb{R}$ s.t. $l \leq a \ \forall a \in A$

There exists — lower bound — For all

Defn $s \in \mathbb{R}$ is the <u>least upper bound</u> (also called <u>supremum</u>) for a set $A \subseteq \mathbb{R}$ if

(i) s is an upper bound for A

(ii) If b is any upper bound for A then $s \leq b$

We write <u>$s = \sup A$</u>

Similarly, greatest lower bound / infimum. $t = \inf A$

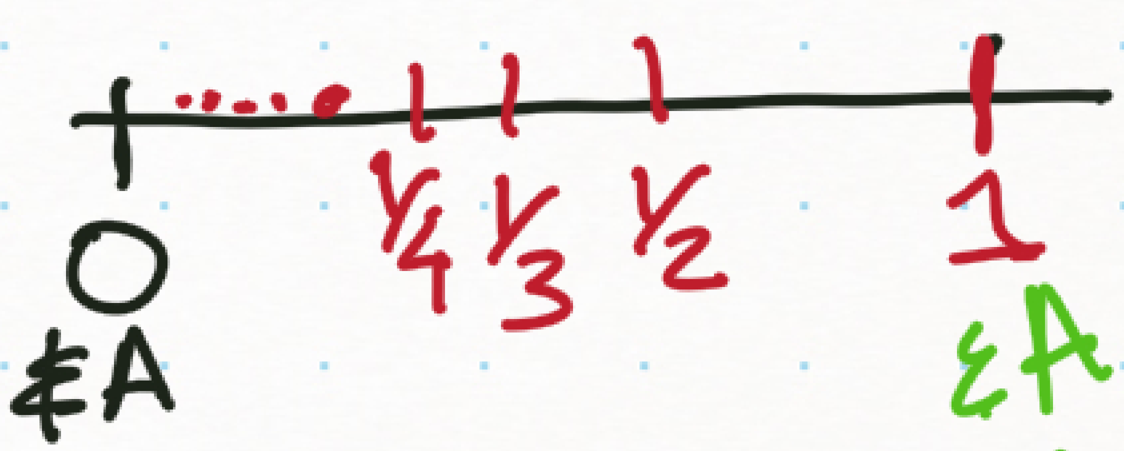
Observation A set can only have one supremum

If $s_1 = \sup A$ & $s_2 = \sup A$ then $s_1 \le s_2$ (by prop(ii) of s_1) prop (i) of s_2

$s_2 \le s_1$ (— " — of s_2)

so, $s_1 = s_2$

Example $A = \{\frac{1}{n} ; n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$



$\sup A = 1$ (i) $\frac{1}{n} \le 1 \ \forall n \in \mathbb{N}$

(ii) consider $b < 1$, then b is not an upper bound because $\underset{\in A}{1} > b$.

$\therefore$ 1 least upper bd.

$\inf A = \ ?$

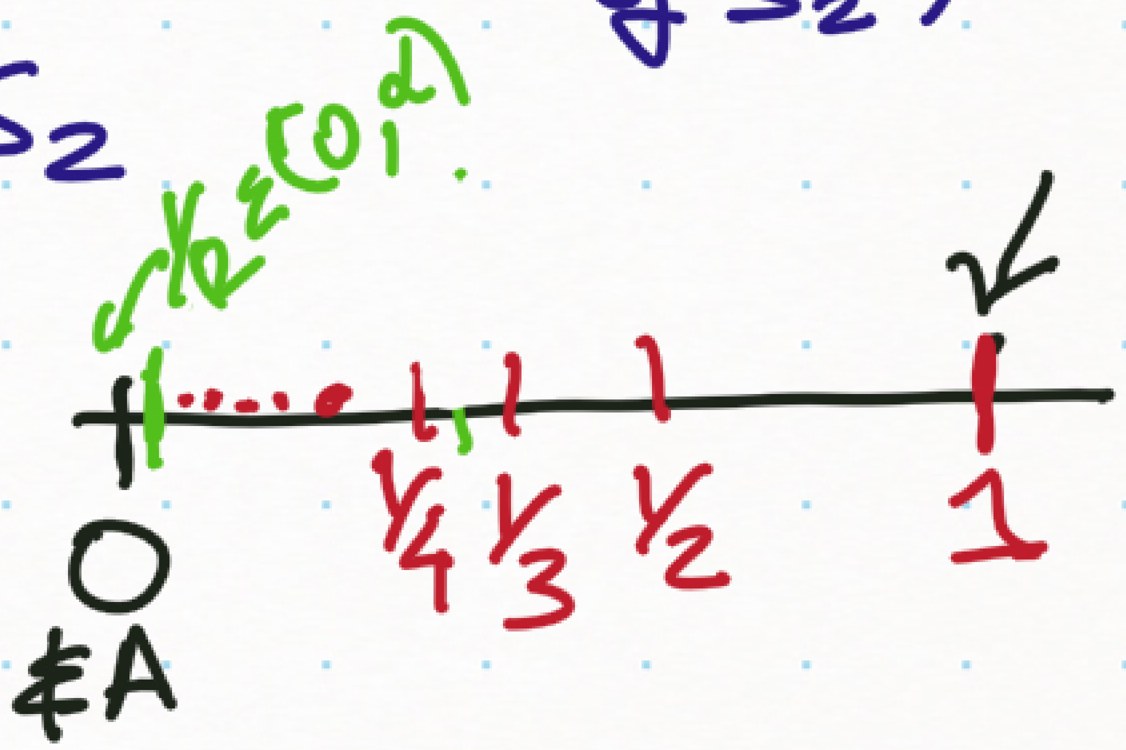
Observation A set can only have one supremum

If $s_1 = \sup A$ & $s_2 = \sup A$ then $s_1 \le s_2$ (by prop (ii) of s_1)

$s_2 \le s_1$ (—"— of s_2)

so, $s_1 = s_2$

Example $A = \left\{\frac{1}{n} ; n \in \mathbb{N}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\}$



$\sup A = 1$

$\inf A = 0$ (i) $0 \le \overset{= 1/n}{a}$ for every $a \in A$ (lower bound)

(ii) It is not possible for $\alpha > 0$ to be a lower bd. of A.

α is not a lower bd of A

What about $\frac{1}{\alpha}$?

$\exists\ k \in \mathbb{N}$ s.t. $k > 1/\alpha \Rightarrow \overset{\in A}{\frac{1}{k}} < \alpha$

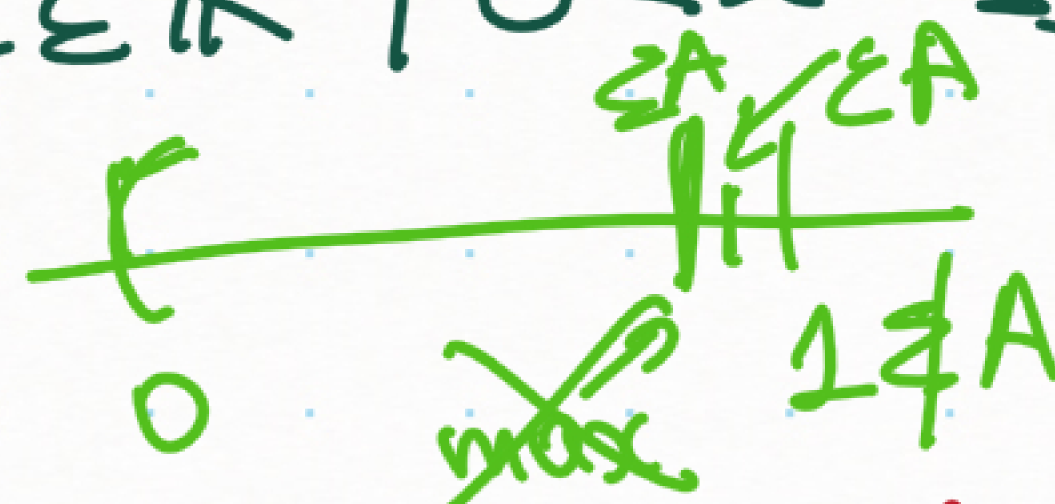
so

Defn $a_0 \in \mathbb{R}$ is a maximum of $A \subseteq \mathbb{R}$ if $a_0 \in A$ and $a_0 \geq a \ \forall a \in A$

$a_1 \in \mathbb{R}$ is a minimum of $A \subseteq \mathbb{R}$ if $a_1 \in A$ and $a_1 \leq a \ \forall a \in A$

e.g. $[0, 1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ $\quad$ max $= 1 =$ sup
min $= 0 =$ inf

$(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ $\quad$ max does not exist $\quad$ sup $= 1$
min does not exist $\quad$ inf $= 0$



Although not every nonempty bounded ~~set~~ in $\mathbb{R}$ contains max, Axiom of completeness tells us that it will always have a sup.

But what about $\mathbb{Q}$?

$\sqrt{2} = 1.4142\ldots\ldots$

<u>example in $\mathbb{Q}$</u> $S = \{q \in \mathbb{Q} : q^2 < 2\}$

Is S bounded above? Yes $10, 5, 2, 3/2 \ldots\ldots$

Does S have a supremum in $\mathbb{Q}$? No. $\boxed{1.42 \text{ sup } \times}$ 1.415 upp. bdd smaller than 1.42

Does S have a supremum in $\mathbb{R}$? Yes $\sqrt{2}$

<u>Thm</u> There exists an unique complete ordered field.
We call this field $\mathbb{R}$.

Try to prove: Let $A \subseteq \mathbb{R}$ be nonempty & bounded above.

(Example 1.3.7) Let $s = \sup A$.

Define $c + A = \{c + a \mid a \in A\}$

Prove that $\sup(c+A) = c + s$.

upper bound of $c+A$? ✓

least upper bound of $c+A$? ✓

Alternate way of thinking about "sup A is the least upper bound"

<u>Lemma</u> Let $s \in \mathbb{R}$ be an upper bound for $A \subseteq \mathbb{R}$.

Then, $s = \sup A \iff$ For each $\epsilon > 0$, $\exists\, a \in A$ s.t. $s - \epsilon < a$

any numbers smaller than s is <u>not</u> an upper bd. of A

$\boxed{\Rightarrow}$ $s = \sup A$

$s - \epsilon < s$

$\Rightarrow s - \epsilon$ is not an upper bound of A since s is the least u. b. of A (prop. (ii) of $s = \sup A$)

$\Rightarrow \exists\, a \in A$ s.t. $\underbrace{s - \epsilon}_{\text{not an u.b.}} < a$.

Alternate way of thinking about "sup A is the least upper bound"

Lemma Let $s \in \mathbb{R}$ be an upper bound for $A \subseteq \mathbb{R}$.

Then, $s = \sup A \iff$ For each $\epsilon > 0$, $\exists\, a \in A$ s.t. $s - \epsilon < a$ ⊛

$\Leftarrow$ verify property (i) of defn. of $s = \sup A$

If b is any number less than s

Then claim b is not an upper bound

$\iff \exists\, a \in A$ s.t. $b < a$

$\Leftarrow$ Take $\epsilon = s - b > 0$

apply ⊛, $\exists\, a \in A$ s.t. $s - (s - b) < a$

$\iff b < a$.

b is not an u.b.

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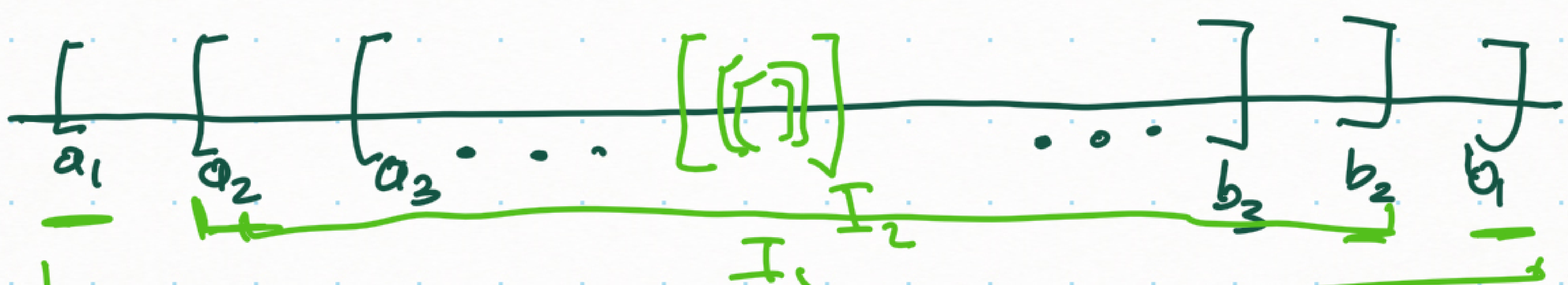
Video #5

Nested Interval Property

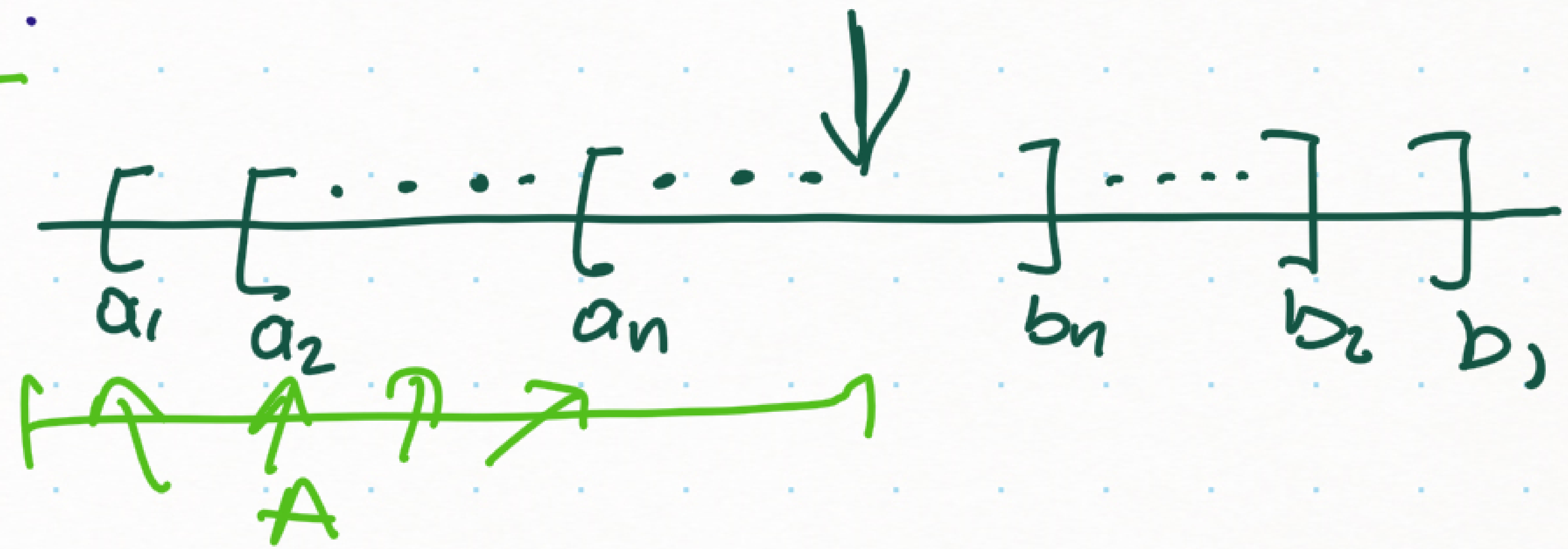
For each $n \in \mathbb{N}$, $I_n = [a_n, b_n] = \{x \in \mathbb{R} \mid a_n \leq x \leq b_n\}$ is a given closed interval s.t. $I_n \supseteq I_{n+1}$ $\forall n = 1, 2, 3, \ldots\ldots$

Then, this nested sequence of closed intervals has a nonempty intersection: $\boxed{\bigcap_{n=1}^{\infty} I_n \neq \emptyset}$

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots\cdots$$



Proof Idea is to use AOC (Axiom of Completeness) to show $\exists\, a \in I_n \ \forall n$.



$A = \{a_n \mid n \in \mathbb{N}\}$

Is A bounded above? Yes, all $b_i \geq a_i \ \forall i$

Let $a = \sup A$ by AoC

Does $a \in I_n = [a_n, b_n]$?

a is an upper bound of A, i.e. $a \geq a_i \ \forall i$

$\Rightarrow a \geq a_n$

a is the least upper bd. of A. & each b_n is an upper bd. of A) $\Rightarrow a \leq b_n$

$\Rightarrow a \in I_n = [a_n, b_n]$ for every n $\Rightarrow a \in \bigcap_{n=1}^{\infty} I_n$ $\blacksquare$

Archimedean Property [$\mathbb{N}$ is unbounded]

① Let $x \in \mathbb{R}$, then $\exists\, n \in \mathbb{N}$ s.t. $n > x$

② Let $y > 0$ in $\mathbb{R}$, then $\exists\, n \in \mathbb{N}$ s.t. $1/n < y$

Proof ① Assume $\mathbb{N}$ is bounded above.
Since $\mathbb{N}$ nonempty subset of $\mathbb{R}$, by AoC, $\mathbb{N}$ must have a sup. Let $\alpha = \sup \mathbb{N}$

Consider $\alpha - 1$ $\leftarrow$ Is this an up. bd. of $\mathbb{N}$? No

$\exists\, n \in \mathbb{N}$ s.t. $\alpha - 1 < n \Rightarrow \underbrace{n+1}_{\in \mathbb{N}} > \alpha \Rightarrow$ α cannot be an u.b. of $\mathbb{N}$.

contradiction.

② Apply ① with $x = 1/y$.

$y > 0 \Rightarrow \frac{1}{y} \in \mathbb{R} \xRightarrow{①} \exists\, n > \frac{1}{y} \Rightarrow y > \frac{1}{n}$.

Theorem [$\mathbb{Q}$ is dense in $\mathbb{R}$]

For every two real numbers $a<b$, $\exists\, r \in \mathbb{Q}$ s.t. $a<r<b$.

Proof We want to find $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ s.t. $\underline{a<\frac{m}{n}<b}$.

By Archimedean Prop., we can pick $n \in \mathbb{N}$ s.t.

$$\underline{\frac{1}{n} < b-a}$$

we know.

$\underline{na} < m < nb$.

We want to choose m s.t. $m > na$
let m be the smallest integer larger than na

i.e., $m-1 \le na < m$

$\Rightarrow a < \frac{m}{n}$ ✓

we know $a < b - \frac{1}{n}$ $\left(\because \frac{1}{n} < b-a\right)$

$\Rightarrow m \le na+1 < n\left(b-\frac{1}{n}\right)+1 = nb-1+1$

$= nb$

$\therefore a < \frac{m}{n} < b$ $\left(\because a < \frac{m}{n} \text{ \& } m < nb\right)$ ▨

(e.g. $na = 2.73$

$m = 3$, $2 \le 2.73 < 3$

$na = 3$

$m = 4$, $3 \le 3 < 4$

<u>Corollary</u> Given any two real numbers $a < b$, $\exists\ t \in \mathbb{I}$ (irrational number) s.t. $a < t < b$

<u>Proof</u> Apply above theorem ($\mathbb{Q}$ dense in $\mathbb{R}$) to $a-\sqrt{2}$ and $b-\sqrt{2}$ $\in \mathbb{R}$.

Think!

MATH 400

Real Analysis

Video #6

Theorem There exists $\alpha \in \mathbb{R}$ s.t. $\alpha^2 = 2$.

Proof [**Recall** the example we did in video #4.
$S = \{r \in \mathbb{Q} \mid r^2 < 2\}$ S was bdd. above
But it didn't have
a least upper bound in $\mathbb{Q}$]

Let $T = \{t \in \mathbb{R} \mid t^2 < 2\}$
set $\alpha = \sup T$ (by AoC)

Claim ① $\alpha^2 < 2$ is not possible
② $\alpha^2 > 2$ is not possible

<u>Claim</u> $\alpha^2 < 2$ is not possible

<u>Assume $\alpha^2 < 2$</u>

[Why not? α is not an upper bound of T

Find $t \in T$ s.t. $\alpha < t$

$$\left(\alpha+\frac{1}{n}\right)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2}$$

want < 2

$$< \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} = \alpha^2 + \frac{2\alpha+1}{n}$$

[Since $\alpha^2 < 2$ we need to make $\boxed{\frac{2\alpha+1}{n}}$ small enough so that $\alpha^2 + \frac{2\alpha+1}{n} < 2$

Choose $n_0 \in \mathbb{N}$ s.t. $\alpha^2 + \frac{2\alpha+1}{n_0} < 2$

i.e., $\frac{1}{n_0} < \frac{2-\alpha^2}{2\alpha+1}$

[possible by Archimedean Property

$$\therefore \left(\alpha+\frac{1}{n_0}\right)^2 < \alpha^2 + \frac{2\alpha+1}{n_0} < 2$$

So, $\alpha + \frac{1}{n_0} \in T$ but $\alpha + \frac{1}{n_0} > \alpha$, an upper bd. of T

contradiction.

Claim $\alpha^2 > 2$ is not possible

Assume $\alpha^2 > 2$

$$\left(\alpha - \frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2}$$
$$> \alpha^2 - \frac{2\alpha}{n}$$

Why not?
α is not the least upper bd. of T.
Find an u.bound b of T s.t. $b < \alpha$

Repeat the idea from previous claim & show that $\alpha - \frac{1}{n_0}$ is an upper bound of T (which is smaller than α).

$\boxtimes$

Ques How would you adapt this proof to show $(x)^{1/m}$ exists for any $x \geq 0$ & $m \in \mathbb{N}$. ?

MATH 400

Real Analysis

Video # 7

How do we count the number of elements in a set?

cardinality of the set

Why are $\{0, 1, 2, 3, \dots\}$ the counting numbers?

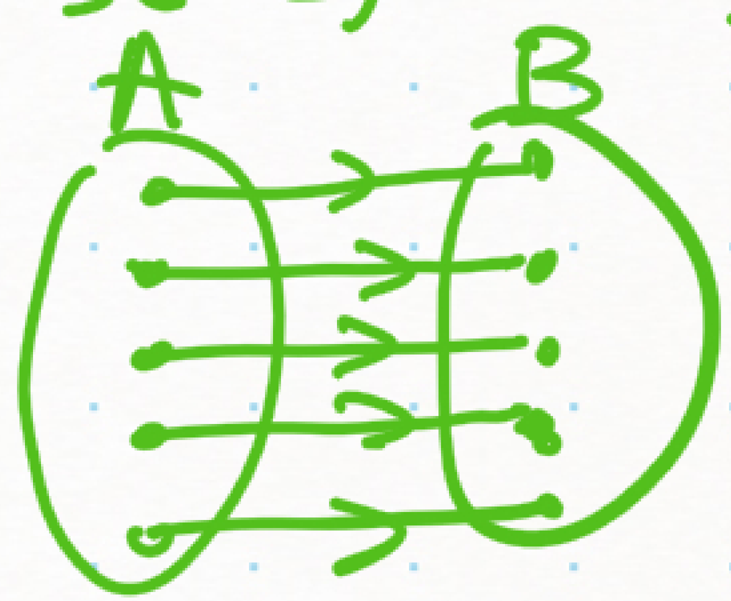
Why do we say there are 3 birds in this?

Established a bijection with $\{1, 2, 3\}$

What happens when a set has infinitely many elements?

Georg Cantor (1845-1918)

<u>Defn</u> A function $f: A \to B$ is a <u>bijection</u> if it is one-to-one ($a_1 \neq a_2$ in $A \Rightarrow f(a_1) \neq f(a_2)$ in B) and onto ($\forall b \in B, \exists a \in A$ s.t. $f(a) = b$)



<u>Defn</u> We say $\underline{A \sim B}$, <u>A has same cardinality as B</u> if $\underline{\exists\ f: A \to B \text{ that is a bijection}}$.

Hilbert's Hotel

Hilbert's hotel has infinitely many rooms in a row.
A Friday evening, the hotel is full (all rooms are occupied)

One new guest arrives → move every guest from their current room to the next room over.
Now, room #1 is empty.

Two new guests arrive →

Infinitely many new guests arrive →

Room #1 $\to$ Room #2
Room #2 $\to$ Room #4
⋮
Room #n $\to$ Room #2n
⋮

All odd numbered rooms are now empty.

Example Let $E = \{2, 4, 6, \cdots\}$ set of evens

$E \subsetneq \mathbb{N}$ but $E \sim \mathbb{N}$

$$\begin{array}{cccccccccc}
\mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & n & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & & & \updownarrow & \\
E: & 2 & 4 & 6 & 8 & 10 & 12 & \cdots & 2n & \cdots
\end{array}$$

$f: \mathbb{N} \to E$ s.t. $f(n) = 2n \quad \forall n \in \mathbb{N}$

Check f is a bijection

Observation Writing all of a set B as a list with no repetitions then this gives us a bijection between $\mathbb{N}$ & B.

$B: b_1 \; b_2 \; b_3 \; b_4 \cdots\cdots$

__Example__ Let $E = \{2, 4, 6, \cdots\}$ set of evens

$E \subsetneqq \mathbb{N}$ but $E \sim \mathbb{N}$. $f : \mathbb{N} \to E$ as $f(n) = 2n$ $\forall n$.

$$\begin{array}{ccccccccc}
\mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & \cdots\cdots & n \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & & \updownarrow \\
E: & 2 & 4 & 6 & 8 & 10 & 12 & \cdots\cdots & 2n \cdots
\end{array}$$

__Example__ $\mathbb{N} \subsetneqq \mathbb{Z}$ but $\mathbb{N} \sim \mathbb{Z}$

$f : \mathbb{N} \to \mathbb{Z}$

$$f(n) = \begin{cases} (n-1)/2 & \text{if } n \text{ odd} \\ -n/2 & \text{if } n \text{ even} \end{cases}$$

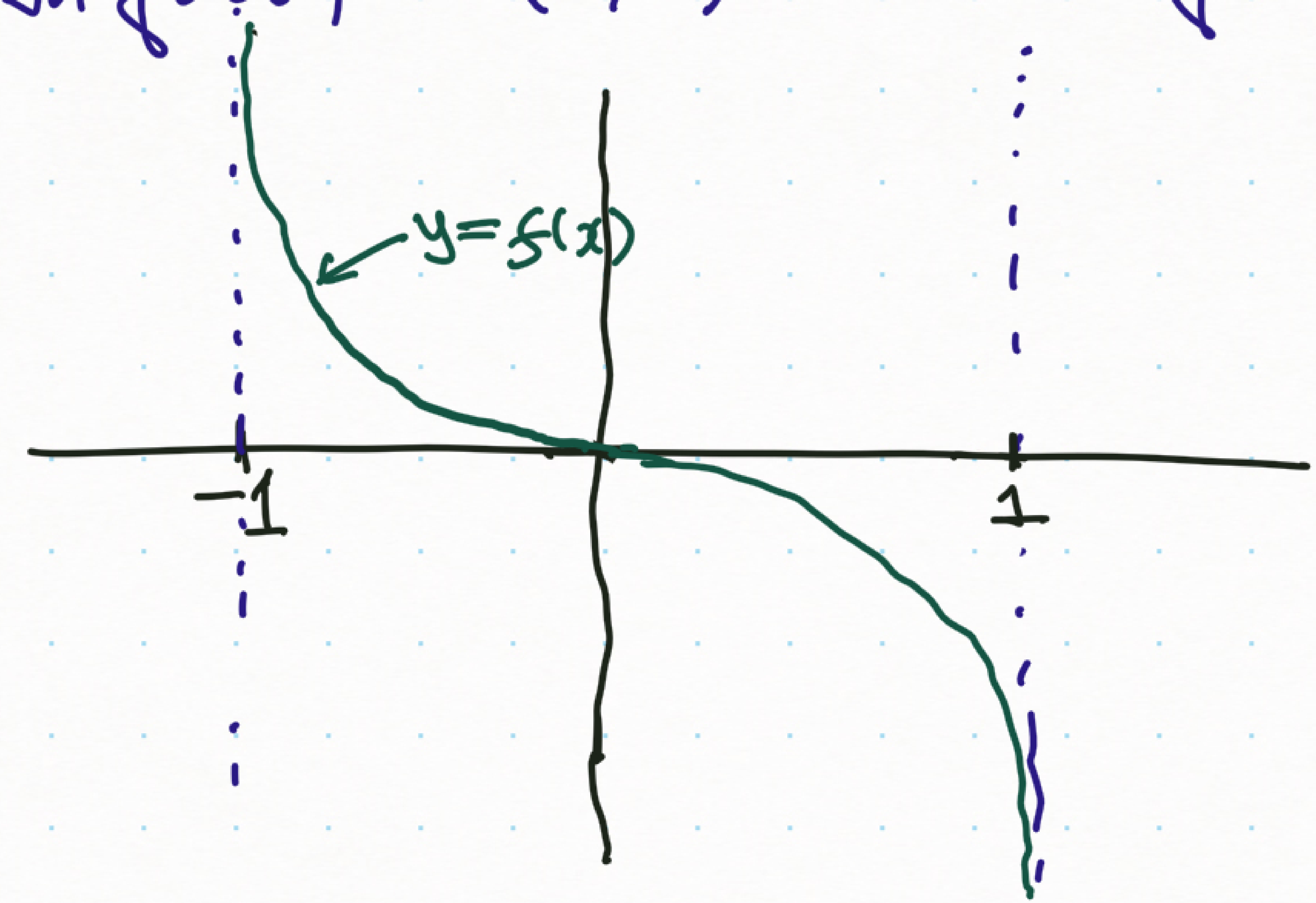
$$\begin{array}{ccccccccc}
\mathbb{N}: & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots & \cdots & \\
\mathbb{Z}: & 0 & -1 & 1 & -2 & 2 & -3 & 3 & \cdots
\end{array}$$

Example Show that $f(x) = \frac{x}{x^2-1}$ gives a bijection between the interval $(-1, 1)$ and $\mathbb{R}$

So, $(-1, 1) \sim \mathbb{R}$

In fact, $(a, b) \sim \mathbb{R}$ for any interval (a, b)

graph of $\xrightarrow{f}$



Defn Set A is <u>countable</u> if $\mathbb{N} \sim A$.

An infinite set that is not countable is called an <u>uncountable set</u>.

Finite sets
Countable sets
Uncountable sets

<u>Are there any uncountable sets</u>? $\cancel{\mathbb{N}}$ $\cancel{\mathbb{Z}}$ $\mathbb{Q}$? $\mathbb{R}$?

Defn Set A is __countable__ if $\mathbb{N} \sim A$.

An infinite set that is not countable is called an __uncountable set__.

__Are there any uncountable sets__? $\cancel{\mathbb{N}}$ $\cancel{\mathbb{Z}}$ $\cancel{\mathbb{Q}}$ $\mathbb{R}$

Theorem $\mathbb{Q}$ is countable

(Even though there are infinitely many rationals between any two integers!!

Proof.

$$
\begin{array}{cccccccccccc}
\mathbb{N} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \ \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow \\
\mathbb{Q} & 0 & \frac{1}{1} & -\frac{1}{1} & \frac{1}{2} & -\frac{1}{2} & \frac{2}{1} & -\frac{2}{1} & \frac{1}{3} & \frac{-1}{3} & \frac{3}{1} & -\frac{3}{1} \ \cdots
\end{array}
$$

Let think of $\mathbb{Q}^+$ first

	1	2	3	4	5	6	7	...
1	$1/1$	$2/1$	$3/1$	$4/1$	$5/1$	$6/1$	$7/1$	...
2	$1/2$	$2/2$	$3/2$	$4/2$	$5/2$	$6/2$	$7/2$	...
3	$1/3$	$2/3$	$3/3$	$4/3$	$5/3$	$6/3$	$7/3$	...
4	$1/4$	$2/4$	$3/4$	$4/4$	$5/4$	$6/4$	$7/4$	...
5	$1/5$	$2/5$	$3/5$	$4/5$	$5/5$	$6/5$	$7/5$	...
6	$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	$6/6$	$7/6$	.....
7	$1/7$	$2/7$	$3/7$	$4/7$	$5/7$	$6/7$	$7/7$	...
⋮								

Start

1/1	2/1	3/1	4/1	5/1	6/1	7/1	...
1/2	2/2	3/2	4/2	5/2	6/2	7/2	...
1/3	2/3	3/3	4/3	5/3	6/3	7/3	...
1/4	2/4	3/4	4/4	5/4	6/4	7/4	...
1/5	2/5	3/5	4/5	5/5	6/5	7/5	...
1/6	2/6	3/6	4/6	5/6	6/6	7/6	...
1/7	2/7	3/7	4/7	5/7	6/7	7/7	...

Each rational is hit more than once: p/q is encountered in positions (p,q), $(2p,2q)$, $(3p,3q)$, ...

<u>Easy Fix</u> → Just skip over a number already encountered

Start

$\boxed{1/1}$	$\boxed{2/1}$	$3/1$	$4/1$	$5/1$	$6/1$	$7/1$	...
$1/2$	$\cancel{2/2}$	$3/2$	$\cancel{4/2}$	$5/2$	$6/2$	$7/2$	...
$1/3$	$2/3$	$\cancel{3/3}$	$4/3$	$5/3$	$\cancel{6/3}$	$7/3$	...
$1/4$	$2/4$	$3/4$	$\cancel{4/4}$	$5/4$	$6/4$	$7/4$	...
$1/5$	$2/5$	$3/5$	$4/5$	$\cancel{5/5}$	$6/5$	$7/5$	...
$1/6$	$2/6$	$3/6$	$4/6$	$5/6$	$\cancel{6/6}$	$7/6$	...
$1/7$	$2/7$	$3/7$	$4/7$	$5/7$	$6/7$	$\cancel{7/7}$	...

It is possible to even write an explicit formula for the bijection just shown!! (Try it!)

Here is another way:

Set $A_1 = \{0\}$, $A_n = \{\pm p/q : p,q \in \mathbb{N}$ in the lowest terms with $p+q = n\}$

$A_1 = \{0\}$, $A_2 = \{\frac{1}{1}, \frac{-1}{1}\}$, $A_3 = \{\frac{1}{2}, \frac{-1}{2}, \frac{2}{1}, \frac{-2}{1}\}$,

$A_4 = \{\frac{1}{3}, \frac{-1}{3}, \frac{3}{1}, \frac{-3}{1}\}$, ...

Each A_n is finite & each rational appears in exactly one of these sets.

Bijection with $\mathbb{N}$: consecutively list the elements of A_1 followed by A_2 followed by A_3 . . . . .

MATH 400

Real Analysis

Video #8

Theorem (Cantor 1874) $\mathbb{R}$ is uncountable

Proof [Cantor's Diagonalization Method (Cantor 1891)]

We prove something stronger

Interval $(0,1)$ is uncountable

(Think: Why is "$(0,1)$ uncountable" $\Leftrightarrow$ "$\mathbb{R}$ uncountable" ?)

$\Uparrow$ derive a bijective function between $(0,1)$ & $\mathbb{R}$

$\Downarrow$ contrapositive

Assume $(0,1)$ is countable

i.e, $\exists$ bijection $f : \mathbb{N} \to (0,1)$

$$
\begin{aligned}
1 &\longleftrightarrow f(1) = .a_{11}\ a_{12}\ a_{13}\ a_{14}\ a_{15} \cdots\cdots \\
2 &\longleftrightarrow f(2) = .a_{21}\ a_{22}\ a_{23}\ a_{24}\ a_{25} \cdots\cdots \\
3 &\longleftrightarrow f(3) = .a_{31}\ a_{32}\ a_{33}\ a_{34}\ a_{35} \cdots\cdots \\
4 &\longleftrightarrow f(4) = .a_{41}\ a_{42}\ a_{43}\ a_{44}\ a_{45} \cdots\cdots \\
5 &\longleftrightarrow f(5) = .a_{51}\ a_{52}\ a_{53}\ a_{54}\ a_{55} \cdots\cdots \\
&\vdots \qquad\quad \vdots \qquad \vdots \qquad\qquad\qquad\qquad\qquad\qquad \ddots
\end{aligned}
$$

list has **all** reals in $(0,1)$

Define $b \in (0,1)$ as $b = 0.b_1 b_2 b_3 b_4 \cdots\cdots$ as $b_n = \begin{cases} 2 & \text{if } a_{nn} \neq 2 \\ 3 & \text{if } a_{nn} = 2 \end{cases}$

claim: b is not in the list above

$\because$ b_i doesnot match with a_{ii} $\forall i$

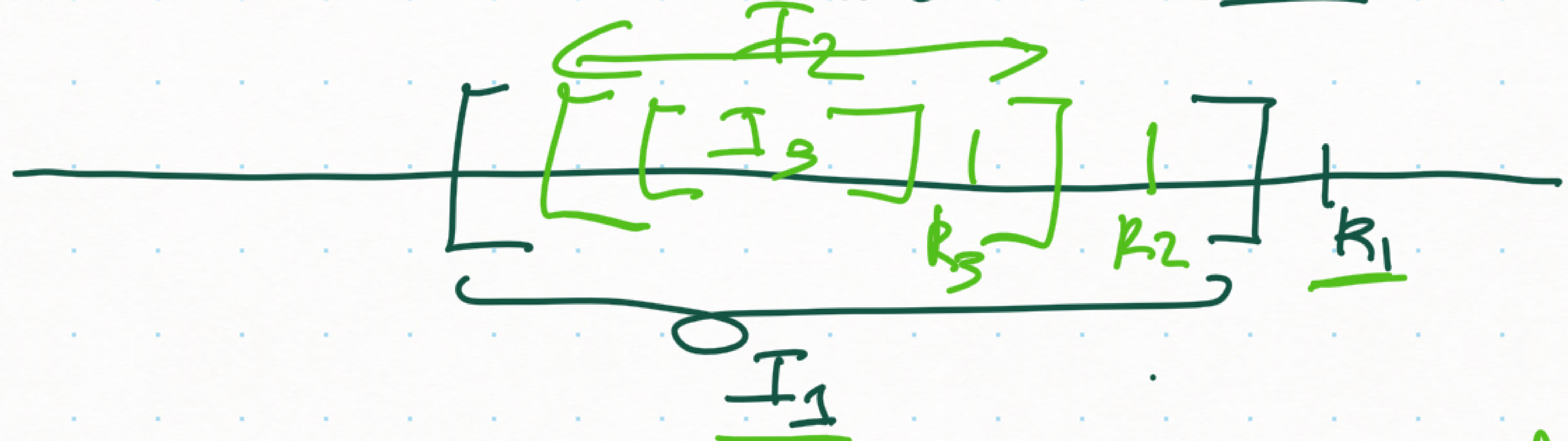
Contradiction

[Proof of "R uncountable" by Cantor (1874)]

Assume $\mathbb{R}$ is countable, so it can be listed as

$$\mathbb{R} = \{r_1, r_2, r_3, r_4, \ldots\}$$

Let I_1 be a closed interval that does not contain r_1.



Let I_2 be a closed interval inside I_1 that does not contain r_2

⋮

Let I_{n+1} be a closed interval inside I_n that does not contain r_n

⋮

We defined $I_1, I_2, I_3, \ldots\ldots$ closed intervals

s.t. $\underline{I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots\ldots \supseteq I_n \supseteq I_{n+1} \supseteq \ldots\ldots}$

and $\underline{r_n \notin I_n \ \forall n}$

Nested Interval Property tells us : $\displaystyle\bigcap_{n=1}^{\infty} I_n \neq \phi$

But, let r be any real number

Then r must equal some r_{n_0} in the list.

But $r \notin I_{n_0}$

which means $r \notin \displaystyle\bigcap_{n=1}^{\infty} I_n$

$\Rightarrow \displaystyle\bigcap_{n=1}^{\infty} I_n = \phi$ <u>contrad.</u>

Properties of countable sets (need proofs)

① If $A \subseteq B$ and B is countable then A is either countable or finite

② If $A_1, A_2, \ldots, A_m$ are each countable sets then $A_1 \cup A_2 \cup A_3 \ldots \cup A_m$ is countable.

③ If A_n is countable for each $n \in \mathbb{N}$ then $\bigcup_{n=1}^{\infty} A_n$ is countable.

"countable union of countable sets"

Cardinalities of Sets

We use the notation $|A|$ to denote the cardinality of set A.

We have shown $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$

we know $|\mathbb{N}| \neq |\mathbb{R}|$ (Cantor)

and $|\mathbb{N}| \leq |\mathbb{R}|$ ($\mathbb{N} \subseteq \mathbb{R}$)

Recall

① $|A| = |B| \iff \exists$ bijective function from A to B

② $|A| \leq |B| \iff \exists$ one-to-one function from A to B

③ $|A| \geq |B| \iff \exists$ onto function from A to B

Infinitely many Infinities

For a set A, the power set $\mathcal{P}(A)$ is the collection of all subsets of A.

$$\mathcal{P}(A) = \{ B \mid B \subseteq A \}$$

(If $|A| = n$ then $|\mathcal{P}(A)| = 2^n$)

e.g. $A = \{1, 2\}$
$\mathcal{P}(A) = \{ \phi, \{1\}, \{2\}, A \}$
$f(1) = \{1\}, f(2) = \{2\}$

Ques Find a 1-1 mapping from A to $\mathcal{P}(A)$

$f : A \to \mathcal{P}(A)$, let $a \in A$, $f(a) = \{a\}$ check 1-1.

Ques What about an onto mapping from A to $\mathcal{P}(A)$

Cantor's Thm Let A be any set. Then
$$|A| < |\mathcal{P}(A)|$$

Proof

Proof by contradiction. Assume $|A| \geq |\mathcal{P}(A)|$

i.e. $\exists f : A \to \mathcal{P}(A)$ that is <u>onto</u>

Every subset of A appears as $f(a)$ for some $a \in A$.

$$B = \{a \in A : a \notin f(a)\}$$

$f(a)$ → subset of A

$f(a)$ may or may <u>not</u> include a

Since f is onto & $B \subseteq A$
then there must $\exists\, b \in A$ s.t. $f(b) = B$

Does $b \in B$? Then $b \notin f(b) = B$ (by defn of B)
Does $b \notin B$? Then $b \in f(b) = B$ (by defn of B)

contradictions

Corollary [$\exists$ infinitely many infinities]

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{N})))| < \cdots$$

$\parallel \longleftarrow$ Think!

$|\mathbb{R}|$

MATH 400

Real Analysis

Video #9

Defn A sequence is a function whose domain is $\mathbb{N}$

$$f: \mathbb{N} \to \mathbb{R}$$

$f(n)$ is the n^{th} term on the list

we often write $f(n)$ as a_n or x_n, etc.

Sometimes, a sequence is indexed to start from $n = n_0$

Most important definition

[Convergence of a sequence]

A sequence (a_n) converges to a real number a if for all $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_n - a| < \epsilon$ $\forall n \geq N$.

↳ small positive real # ϵ

↳ term of seq. after which a_n is close to a

↳ distance between a_n and a is atmost ϵ

↳ for all n that are large enough

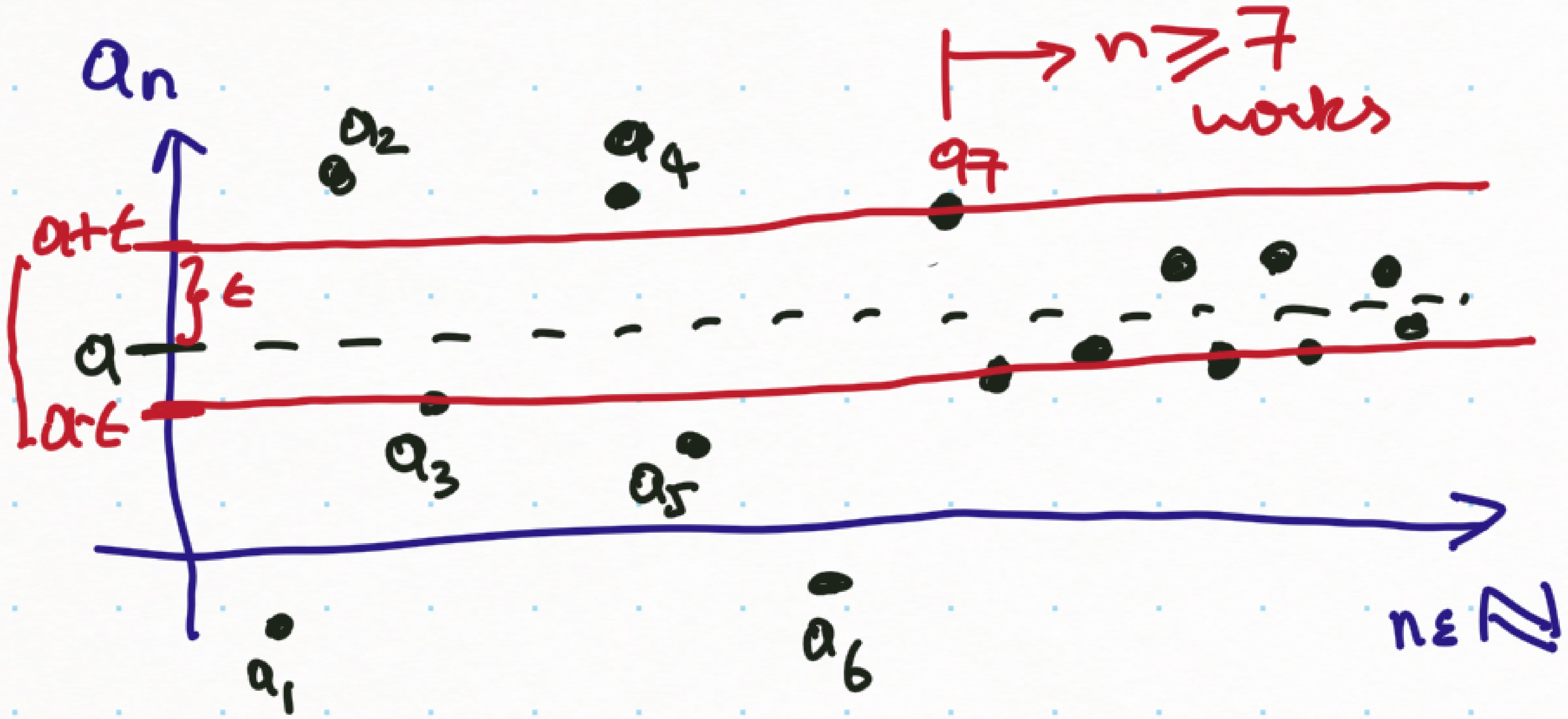
$a_n \to a$ as $n \to \infty$ if $\forall \epsilon > 0, \exists N$ s.t. $|a_n - a| < \epsilon \ \forall n \geq N$

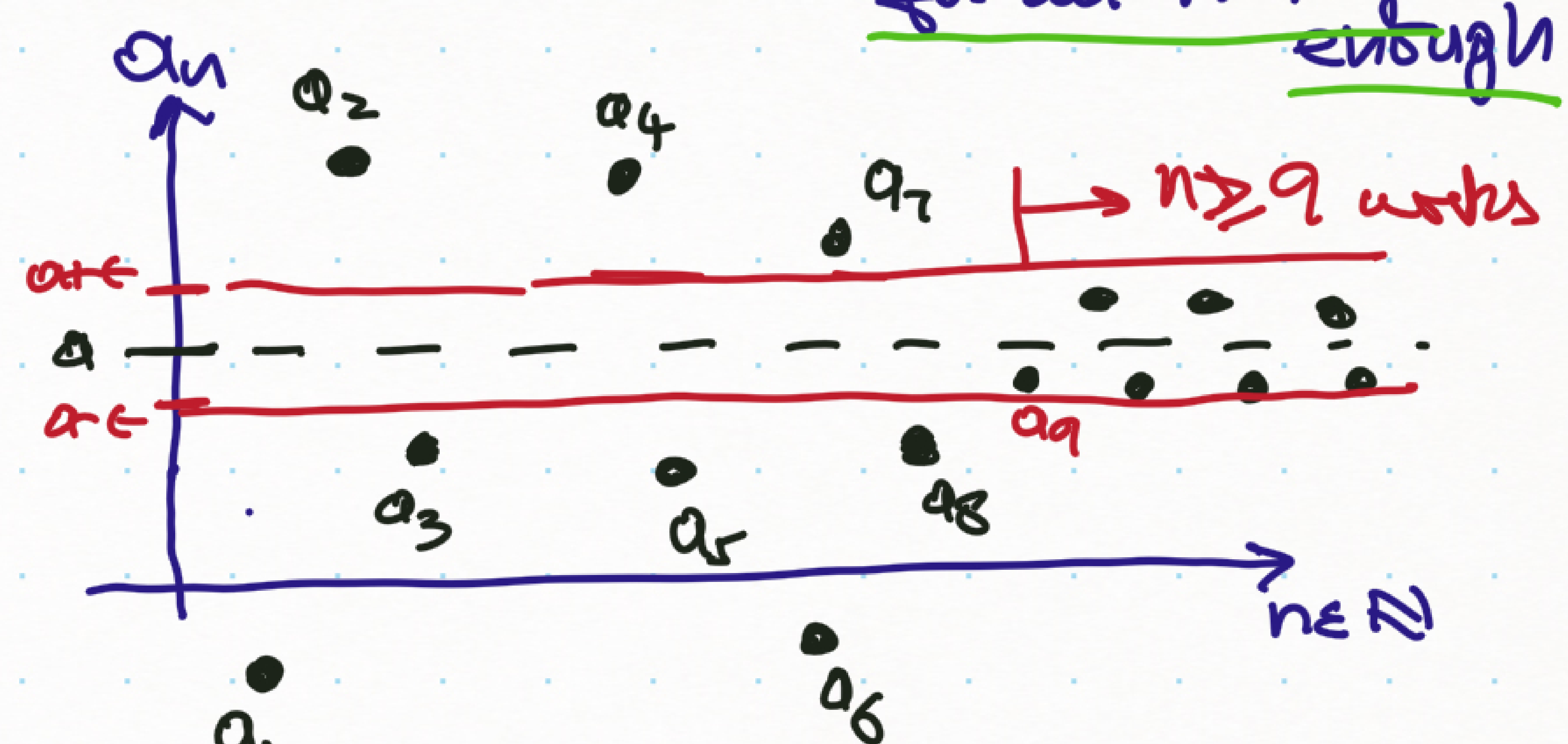
Motivation Recall we proved: $a = b \iff |a-b| < \epsilon \ \forall \epsilon > 0$

a equals b $\iff$ distance between a & b can be made arbitrarily small

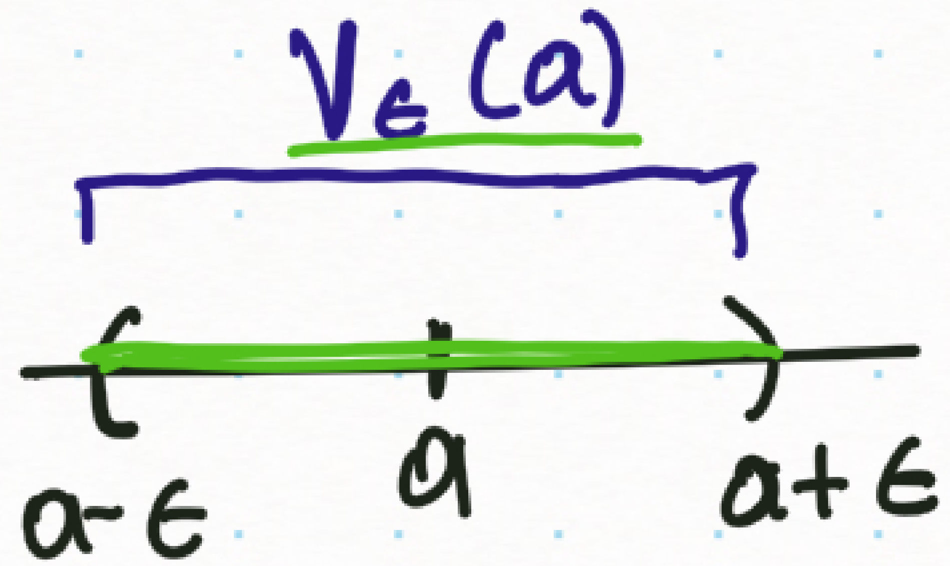
Comment $|a_n - a| < \epsilon \iff -\epsilon < a_n - a < \epsilon$

$\iff a - \epsilon < a_n < a + \epsilon \iff a_n \in (a-\epsilon, a+\epsilon)$ for all n large enough





Defn let $a \in \mathbb{R}$ and $\epsilon > 0$, the set $V_\epsilon(a) = \{x \in \mathbb{R} : |x-a| < \epsilon\}$ is called the <u>ϵ-neighborhood of a.</u>



Defn (<u>Convergence of a sequence: Topological version</u>)

A seq (a_n) converges to a if
given any ϵ-neighborhood $V_\epsilon(a)$ of a
there exists a point in the sequence
after which all of the terms are in $V_\epsilon(a)$.

<u>Remember</u> value of N depends on the choice of ϵ.

Template for proof $a_n \to a$

Step 0 Scratch work: Start with $|a_n - a| < \epsilon$
& unravel to solve for n
(in terms of ϵ)
This will tell us which N to choose.

Actual Proof

Step 1 Let $\epsilon > 0$

Step 2 Let $n > N =$ (where value for N comes from step 0)

Step 3 Redo the scratch work (without ϵ's)
but use the value of N to show
$n \geq N \Rightarrow |a_n - a| < \epsilon$

<u>example</u> Let $a_n = \frac{1}{n}$ for all n $(1, 1/2, 1/3, \ldots\ldots)$

Show $a_n \longrightarrow 0$ as $n \to \infty$.

[scratch work: we want $\boxed{|a_n - a| < \epsilon}$

i.e, $\left|\frac{1}{n} - 0\right| < \epsilon$

i.e., $\frac{1}{n} < \epsilon$, i.e. $\boxed{n > 1/\epsilon}$]

$\therefore$ choose $N = 1/\epsilon$

<u>solution</u> Let $\epsilon > 0$

Set $N = 1/\epsilon$

For any $n > N$,

$$\rightarrow \quad |a_n - a| = \left|\frac{1}{n} - 0\right| = \left|\frac{1}{n}\right| = \frac{1}{n} < \frac{1}{N} = \frac{1}{1/\epsilon} = \epsilon$$

since $n > N$

$\therefore |a_n - a| < \epsilon$.

example Let $a_n = \dfrac{3n+1}{n+2}$. Prove $\lim_{n\to\infty} a_n = 3$

[scratch work: $|a_n - a| < \epsilon \iff \left|\dfrac{3n+1}{n+2} - 3\right| < \epsilon$

$\iff \left|\dfrac{3n+1}{n+2} - \dfrac{3(n+2)}{n+2}\right| < \epsilon \iff \left|\dfrac{3n+1-3n-6}{n+2}\right| < \epsilon$

$\iff \left|\dfrac{-5}{n+2}\right| < \epsilon \iff \dfrac{5}{n+2} < \epsilon \iff \dfrac{5}{\epsilon} < n+2$

$\iff \boxed{n > \dfrac{5}{\epsilon} - 2}$]

Solution Let $\epsilon > 0$

Set $N = \dfrac{5}{\epsilon} - 2$

[$n > N \Rightarrow n+2 > N+2$
$\Rightarrow \dfrac{1}{n+2} < \dfrac{1}{N+2} \Rightarrow \dfrac{5}{n+2} < \dfrac{5}{N+2}$]

For any $n > N$,

$|a_n - a| = \left|\dfrac{3n+1}{n+2} - 3\right| = \cdots\cdots = \left|\dfrac{-5}{n+2}\right| = \dfrac{5}{n+2} < \dfrac{5}{N+2}$

$\dfrac{5}{N+2} = \dfrac{5}{(\frac{5}{\epsilon}-2)+2} = \dfrac{5}{5/\epsilon} = \epsilon$

$\therefore |a_n - a| < \epsilon$.

Math 400

Real Analysis

Video #10

Theorem [Uniqueness of Limits]

The limit of a sequence, when it exists, is unique.

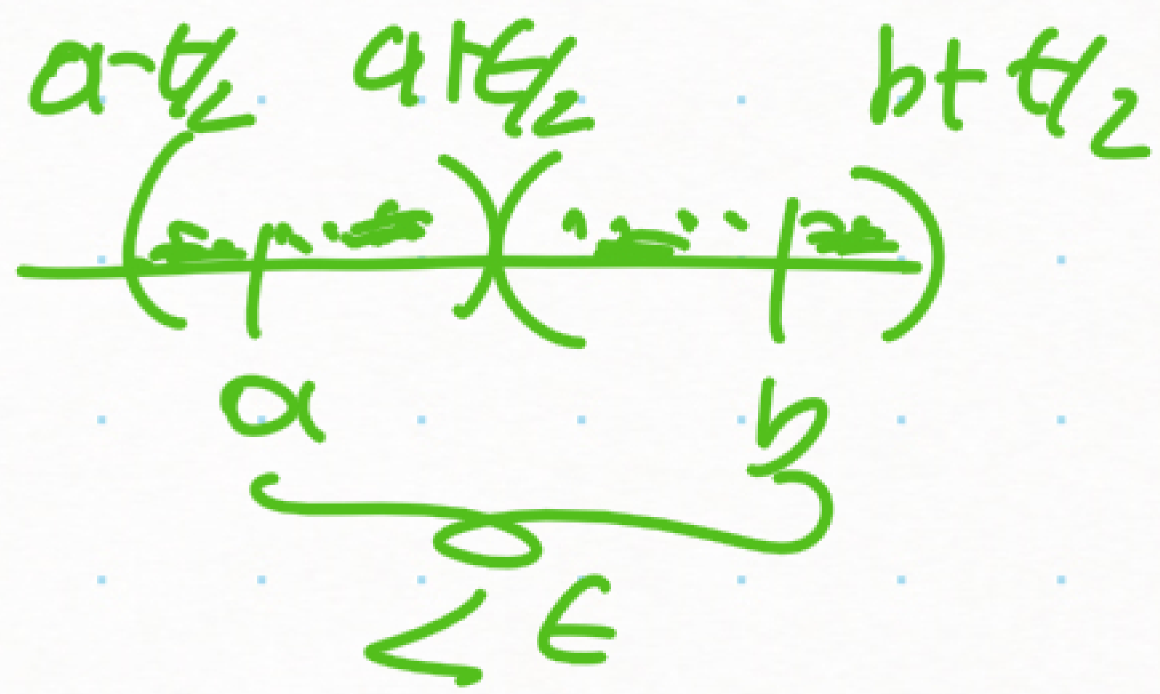
Proof Idea suppose $a_n \to a$ & $a_n \to b$

we want to show $a = b$

by $|a-b| < \epsilon \ \forall \epsilon > 0$

How? as a_n gets closer to a it will be within distance $\epsilon/2$ of a

as a_n gets closer to b it will be within distance $\epsilon/2$ of b



Use Triangle inequality.

Theorem [Uniqueness of Limits]

The limit of a sequence, when it exists, is unique.

Proof Suppose $a_n \to a$ and $a_n \to b$

Let $\epsilon > 0$.

Since $\frac{\epsilon}{2} > 0$ and $a_n \to a$, $\exists N_1$ s.t. $|a_n - a| < \frac{\epsilon}{2}$ $\forall n > N_1$

Since $\frac{\epsilon}{2} > 0$ and $a_n \to b$, $\exists N_2$ s.t. $|a_n - b| < \frac{\epsilon}{2}$ $\forall n > N_2$

Let $n > N = ?$

$$
\begin{aligned}
|a-b| = |a - a_n + a_n - b| &\leq ? \\
&= ? \\
&< ? \\
&= \epsilon
\end{aligned}
$$

Fill in the blanks & discuss in class

Since $|a-b| < \epsilon$ for all $\epsilon > 0$, we have $a = b$ ∎

<u>Defn</u> A sequence that does not converge is said to <u>diverge</u>.

<u>Three forms of divergence</u>

- <u>a_n diverges to ∞</u> if $\forall M > 0$, there exists N such that $a_n > M$ $\forall n > N$.

- <u>a_n diverges to $-\infty$</u> if $\forall M < 0$, there exist N such that $a_n < M$ $\forall n > N$.

- <u>Otherwise, (a_n)'s limit does not exist.</u>

example $a_n = n^2$ Show $\lim_{n\to\infty} a_n = \infty$

Scratch work: we want $a_n > M$

i.e, $n^2 > M$, i.e., $n > \sqrt{M}$

solution Let $M > 0$.

Set $N = \sqrt{M}$

Then, for any $n > N$,

$$a_n = n^2 > N^2 = (\sqrt{M})^2 = M.$$

So, we have shown $a_n > M \ \forall n > N$.

Comment a_n diverges is same as $a_n \not\to a$ for any $a \in \mathbb{R}$.

What is the negation of definition of $a_n \to a$?

$\forall \epsilon > 0, \ \exists N$ s.t. $|a_n - a| < \epsilon \ \forall n > N$.

negation

$\exists \epsilon > 0, \ \forall N \ \exists n > N$ s.t. $|a_n - a| \geq \epsilon$.

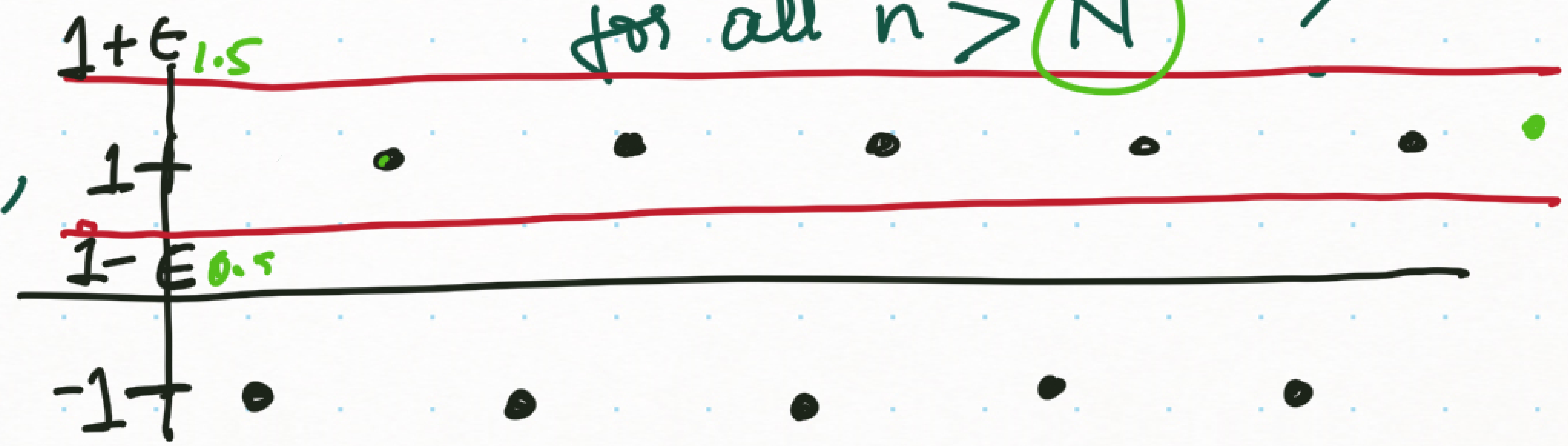
Find such a "bad" $\epsilon > 0$

example Let $a_n = (-1)^n$. Prove (a_n) diverges.

$(-1, 1, -1, 1, -1, 1, \ldots\ldots)$

[scratch work: what ϵ should we choose so that a_n is not within $(a-\epsilon, a+\epsilon)$ for all $n > N$?

Look at $a = 1$,



$\epsilon = \frac{1}{2}$ works

example Let $a_n = (-1)^n$. Prove (a_n) diverges.

Solution Suppose $a_n \to a$

Let $\epsilon = 1/2$

Since $a_n \to a$, there must exist N s.t. $|a_n - a| < \frac{1}{2}$ $\forall n > N$

i.e., $\boxed{|(-1)^n - a| < \frac{1}{2} \ \forall n > N.}$

case 1 n even: For $n > N$, we have

$|1 - a| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < 1 - a < \frac{1}{2} \Leftrightarrow -\frac{3}{2} < -a < -\frac{1}{2} \Leftrightarrow \frac{1}{2} < a < \frac{3}{2}$

case 2 n odd: For $n > N$, we have

$|-1 - a| < \frac{1}{2} \Leftrightarrow -\frac{1}{2} < -1 - a < \frac{1}{2} \Leftrightarrow \frac{1}{2} < -a < \frac{3}{2} \Leftrightarrow -\frac{3}{2} < a < -\frac{1}{2}$

we need $a \in (\frac{1}{2}, 3/2)$ & $a \in (-\frac{3}{2}, -\frac{1}{2})$ Not possible $\times$

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Real Analysis

Video #11

- Why study formal definitions?
- <u>Behavior of convergent sequences</u>

- Why study formal definitions?
- <u>Behavior of convergent sequences</u>

<u>Defn</u> A sequence (x_n) is <u>bounded</u> if
$\exists M > 0$ s.t. $|x_n| < M \ \forall n \in \mathbb{N}$

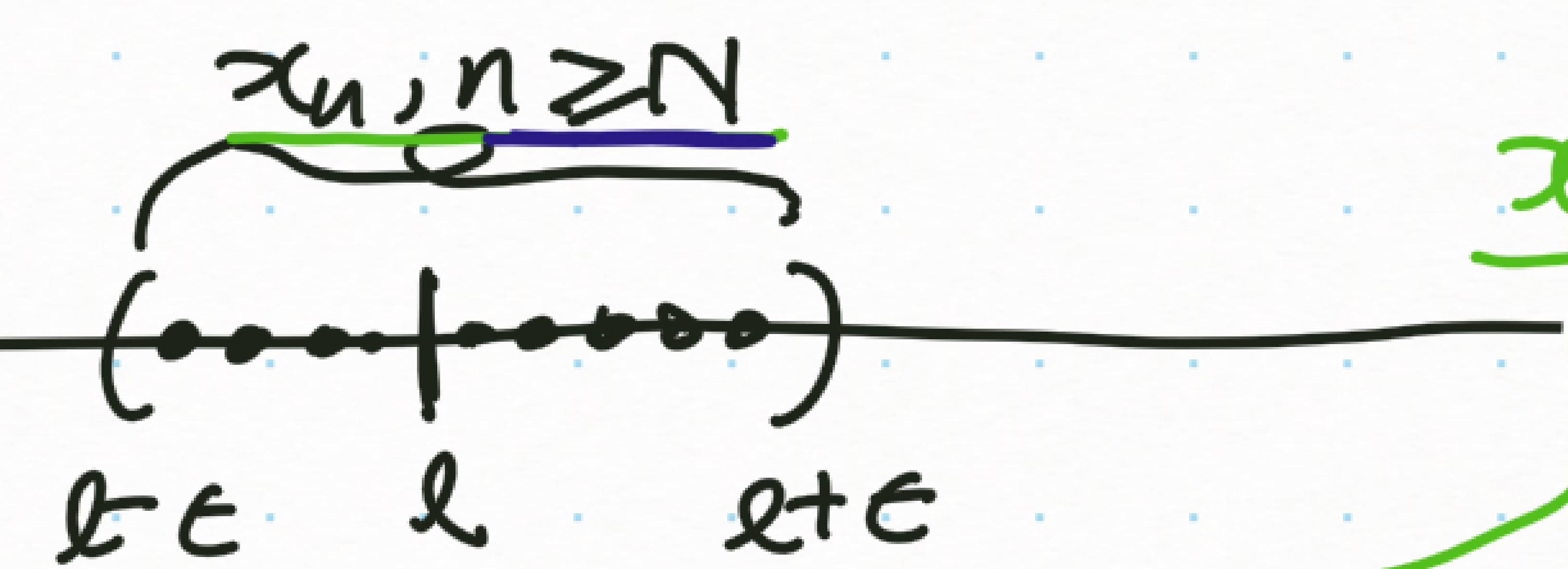
$-M < x_n < M$

This means every $x_n \in [-M, M]$

<u>Theorem</u> Every convergent sequence is bounded.

What does the contrapositive tell us?

If (a_n) is unbdd. then (a_n) is divergent



$x_n \to l$

Theorem Every convergent sequence is bounded.

Proof Suppose $x_n \to l$

For every $\epsilon > 0$, say $\epsilon = 1$, $\exists\, N \in \mathbb{N}$ s.t.

$x_n \in (l-1, l+1)\ \forall n \geq N$

To avoid considering whether l is positive or negative, we can simply use the upper bound:

$|x_n| < |l| + 1 .\ \forall n \geq N.$

But, what about $x_1, x_2, \ldots, x_{N-1}$?

$|x_N|, |x_{N+1}|, \ldots$
$< |l| + 1$ ✓

Let $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |l|+1\}$

$\Rightarrow |x_1| \leq M,\ |x_2| \leq M,\ \ldots,\ |x_{N-1}| \leq M,\ |x_n| \leq M\ \forall n \geq N$

$\Rightarrow |x_n| \leq M\ \forall n \in \mathbb{N}.$

Theorem [Algebra of limits] Let $\lim a_n = a$ & $\lim b_n = b$.

① $\lim (ca_n) = ca$ for all $c \in \mathbb{R}$

② $\lim (a_n + b_n) = a + b$

③ $\lim (a_n b_n) = ab$

④ $\lim (a_n / b_n) = a/b$, when $b \neq 0$.

Proof ① Try it! Straightforward using $\underbrace{|ca_n - ca|}_{\to \text{ small } < \epsilon}$

$$= \underbrace{|c|}_{\text{fixed number}} \underbrace{|a_n - a|}_{\text{small since } a_n \to a}$$

Proof ② Try it! Straightforward using

$$\underbrace{\boxed{|(a_n + b_n) - (a+b)|}}_{< \epsilon} = |(a_n - a) + (b_n - b)| \quad \Delta \text{ ineq.}$$

$$\leq \underbrace{|a_n - a|}_{\text{small } \epsilon/2} + \underbrace{|b_n - b|}_{\text{small } \epsilon/2} < \epsilon$$

Look up details in the textbook.

Proof ④ $\left[\lim \frac{a_n}{b_n} = \frac{a}{b} \text{ if } b \neq 0\right]$

If we can show that $\boxed{b_n \to b \Rightarrow \frac{1}{b_n} \to \frac{1}{b}}$ when $b \neq 0$

Then using ③ we have $a_n \frac{1}{b_n} \to a \frac{1}{b}$ $\quad \frac{a_n}{b_n} \to \frac{a}{b}$

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \left|\frac{b-b_n}{b\,b_n}\right| = \frac{|b-b_n|}{|b|\,|b_n|} \quad \text{want} < \epsilon$$

want $< \epsilon$ for $n \geq N$

$|b-b_n|$ small since $b_n \to b$

$|b|\,|b_n|$ fixed number

why would $|b_n| \geq \delta > 0$?

$|b_n| \to |b| \neq 0$

$\forall n \geq N$

0 $\quad |b|$

Ultimately, $|b_n|$ is going to be close to $|b| > 0$.

$$\frac{1}{|b|\,|b_n|} < \text{fixed number}$$

i.e., $\frac{1}{|b_n|} <$ fixed number

i.e, $\boxed{|b_n| \geq \delta > 0}$ lower bd.

~~$|b_n| < M$~~

~~$\Rightarrow \frac{1}{|b_n|} > \frac{1}{M}$~~

~~we don't that~~

Proof of ③ [$a_n b_n \to ab$]

Scratch work We want to find N s.T. $|a_n b_n - ab| < \epsilon$ $\forall n \geq N$.

we know $a_n \to a$, i.e., $\forall \epsilon_1 > 0$, $\exists N_1$ s.t. $|a_n - a| < \epsilon_1$ $\forall n \geq N_1$

we know $b_n \to b$, i.e., $\forall \epsilon_2 > 0$, $\exists N_2$ s.t. $|b_n - b| < \epsilon_2$ $\forall n \geq N_2$

$|a_n b_n - ab| = |a_n b_n - ? + ? \quad - ab|$

Proof of ③ $[a_n b_n \to ab]$

Scratch work We want to find N s.t. $|a_n b_n - ab| < \epsilon$ $\forall n \geq N$.

we know $a_n \to a$, i.e., $\forall \epsilon_1 > 0$, $\exists N_1$ s.t. $|a_n - a| < \epsilon_1$ $\forall n \geq N_1$

we know $b_n \to b$, i.e., $\forall \epsilon_2 > 0$, $\exists N_2$ s.t. $|b_n - b| < \epsilon_2$ $\forall n \geq N_2$

$$\underbrace{|a_n b_n - ab|}_{\text{small } < \epsilon?} = |a_n b_n \underline{- ab_n + ab_n} - ab|$$

$$\leq |a_n b_n - ab_n| + |ab_n - ab| \qquad \triangle \text{ ineq.}$$

$$= |a_n - a|\,|b_n| + |a|\,|b_n - b|$$

we can make this small since $a_n \to a$

$|a_n - a| < \epsilon_1 = \frac{\epsilon}{2C}$

b_n is bdd.

$|b_n| \leq C$

$\frac{\epsilon}{2C} C = \epsilon/2$

fixed

$b_n \to b$, we can make this small

$|b_n - b| < \epsilon_2 = \frac{\epsilon}{2|a|}$

$|a| \frac{\epsilon}{2|a|} = \epsilon/2$

Proof of ③. Let $\epsilon > 0$

Since (b_n) converges, we know (b_n) is bounded
i.e., $\exists C$ s.t. $|b_n| \leq C \ \forall n$.

Let $\epsilon_1 = \dfrac{\epsilon}{2C+1}$. Since $\epsilon_1 > 0$, $\exists N_1$ s.t. $|a_n - a| < \epsilon_1 \ \forall n \geq N_1$

Let $\epsilon_2 = \dfrac{\epsilon}{2|a|+1}$. Since $\epsilon_2 > 0$, $\exists N_2$ s.t. $|b_n - b| < \epsilon_2 \ \forall n \geq N_2$

Let $n > N = \max\{N_1, N_2\}$; then

$$|a_n b_n - ab| = |a_n b_n - a b_n + a b_n - ab|$$
$$\leq |a_n b_n - a b_n| + |a b_n - ab|$$
$$= |a_n - a|\,|b_n| + |a|\,|b_n - b|$$
$$< \epsilon_1 C + |a|\,\epsilon_2 = \frac{\epsilon}{2C+1} C + |a| \frac{\epsilon}{2|a|+1}$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

$n \geq N_1$ and $n \geq N_2$

$\blacksquare$

<u>example</u> $\lim_{n\to\infty} \frac{1}{2}\left(\frac{1/n + 1/n^2 + 4}{5 - 1/n^2}\right)\left(\frac{3n+1}{n+2} + \frac{1}{\sqrt{n}}\right) = \,??$

<u>we know</u> $\left(1/n\right) \to 0$, $\left(1/n^2\right) \to 0$,

$\left(5 - 1/n^2\right) \to 0$, $\left(\frac{3n+1}{n+2}\right) \to 3$, $\left(\frac{1}{\sqrt{n}}\right) \to 0$

$$\Rightarrow \lim_{n\to\infty} \frac{1}{2}\left(\frac{1/n + 1/n^2 + 4}{5 - 1/n^2}\right)\left(\frac{3n+1}{n+2} + \frac{1}{\sqrt{n}}\right) = \frac{1}{2}\left(\frac{0+0+4}{5}\right)(3+0)$$

$$= \frac{6}{5}.$$

Theorem [Order Limit Theorem] Let $a_n \to a$ & $b_n \to b$.

① If $a_n \geq 0$ $\forall n \in \mathbb{N}$ then $a \geq 0$

② If $a_n \leq b_n$ $\forall n \in \mathbb{N}$ then $a \leq b$

③ If $\exists c \in \mathbb{R}$ s.t. $c \leq b_n$ $\forall n$, then $c \leq b$.
If $\exists d \in \mathbb{R}$ s.t. $a_n \leq d$ $\forall n$, then $a \leq d$.

Theorem [Order Limit Theorem] Let $a_n \to a$ & $b_n \to b$.

① If $a_n \geq 0$ $\forall n \in \mathbb{N}$ then $a \geq 0$

② If $a_n \leq b_n$ $\forall n \in \mathbb{N}$ then $a \leq b$

③ If $\exists c \in \mathbb{R}$ s.t. $c \leq b_n$ $\forall n$, then $c \leq b$.
If $\exists d \in \mathbb{R}$ s.t. $a_n \leq d$ $\forall n$, then $a \leq c$.

Proof ① Think about it!

② $(b_n - a_n) \longrightarrow b - a$ (by algebra of limits)
& since $b_n - a_n \geq 0$, part ① $\Rightarrow$ $b - a \geq 0$, i.e., $b \geq a$.

③ Let $a_n = c$ $\forall n$ & apply part ②.
Let $b_n = d$ $\forall n$ & apply part ②.

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Real Analysis

Video # 12

We know: Convergent sequences are bounded.
But not all bounded seq.s are convergent.
(Recall examples)

Ques Under what condition is a bdd. seq. convergent.

Defn A sequence (a_n) is **increasing** if $a_n \leq a_{n+1}$ $\forall n$
and **decreasing** if $a_n \geq a_{n+1}$ $\forall n$
A sequence is **monotone** if it is either increasing or decreasing.

Monotone Convergence Theorem

If a sequence is monotone and bounded, then it converges

MCT If a sequence is monotone and bounded then it converges.

Proof Let (a_n) be montone and bounded.

Let us assume (a_n) is increasing (proof for (a_n) decreasing is similar)

Consider the set of points $\{a_n : n \in \mathbb{N}\}$

By assumption, this set is bounded ($\because$ (a_n) is bounded)

Where do you think (a_n) converges to?

MCT If a sequence is monotone and bounded then it converges.

Proof Let (a_n) be montone and bounded.

Let us assume (a_n) is increasing (proof for (a_n) decreasing is similar)

Consider the set of points $\{a_n : n \in \mathbb{N}\}$

By assumption, this set is bounded ($\because$ (a_n) is bounded)

$$\text{Let } s = \sup\{a_n : n \in \mathbb{N}\}$$

Claim $\lim_{n\to\infty} a_n = s$

Proof Let $\epsilon > 0$. Since s is the least u.b., $s-\epsilon$ is not an u.b.

$\therefore \exists\, a_N$ s.t. $s-\epsilon < a_N$

Since (a_n) is increasing, $a_N \leq a_n$ $\forall n \geq N$. $\therefore$ $s-\epsilon < a_N \leq a_n \leq s < s+\epsilon$

i.e. $|a_n - s| < \epsilon$ $\forall n \geq N$

What are the different behaviors of monotone sequences?

e.g. $(n) = (1, 2, 3, 4, \dots)$ is an inc. seq. that diverges to ∞

$\left(\frac{1}{n}\right) = \left(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\right)$ is a dec. seq. that converges to 0

Corollary (to MCT) Suppose (a_n) is monotone. Then

$$(a_n) \text{ converges} \iff (a_n) \text{ is bounded}$$

Moreover,

- If (a_n) is increasing, then either (a_n) diverges to ∞ or $\lim_{n\to\infty} a_n = \sup\{a_n : n \in \mathbb{N}\}$
- If (a_n) is decreasing, then either (a_n) diverges to $-\infty$ or $\lim_{n\to\infty} a_n = \inf\{a_n : n \in \mathbb{N}\}$

<u>example</u> Let (a_n) be the sequence defined as

$a_1 = 0.1$, $a_2 = 0.12$, $a_3 = 0.123$, $a_4 = 0.1234$, ...,

$a_{11} = 0.1234567891011$, $a_{12} = 0.123456789101112$, ...

Prove that (a_n) converges.

<u>Proof</u> a_{n+1} and a_n match exactly until the last digits of a_{n+1}, which are the digits of $n+1$.

$a_{n+1} - a_n$ is number $0.\underbrace{00\ldots0}$ $\underline{\text{digits of } n+1} > 0$

i.e, $a_{n+1} - a_n > 0 \Leftrightarrow a_{n+1} > a_n \ \forall n$ $\therefore (a_n)$ inc. seq.

$0.1 \leq a_n \leq 0.2 \ \forall n$ $\therefore (a_n)$ is bdd.

By MCT (a_n) is convergent.