

Math 400

Real Analysis

Video # 13

A Detour to Series

Defn Let (b_n) be a sequence.

An infinite series is a formal expression

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \dots$$

The corresponding sequence of partial sums (s_m) is defined to be $s_m = b_1 + b_2 + \dots + b_m$, $\forall m$.

We say $\sum_{n=1}^{\infty} b_n$ converges to a number B

if (s_m) converges to B

example Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$

The seq. of partial sums (S_m) is $S_m = \underline{1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{m^2}}$

since each term of the series is positive, (S_m) is increasing.

$$S_1 = 1, S_2 = 1 + \frac{1}{4}, S_3 = 1 + \frac{1}{4} + \frac{1}{9}, \dots$$

Is (S_m) bounded?

$$S_m = 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \dots + \frac{1}{m \cdot m}$$

$< M \quad \forall m$

example Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

The seq. of partial sums (S_m) is $S_m = 1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{m^2}$

Since each term of the series is positive, (S_m) is increasing.

Is (S_m) bounded?

$$S_m = 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + \frac{1}{4 \cdot 4} + \dots + \frac{1}{m \cdot m}$$

$$< 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3} + \dots + \frac{1}{m \cdot (m-1)}$$

$$= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{(m-1)} - \frac{1}{m}\right)$$

$$= 1 + 1 - \frac{1}{m} = 2 - \frac{1}{m}$$

$$< 2 \quad \forall m.$$

\therefore By MCT (S_m) converges.

Example (Harmonic Series)

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

Since each term is positive, seq. of partial sums (S_m) is increasing.

$$S_m = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}$$

Is (S_m) bounded?

Is 2 an upper bound?

Is 2.5 an upper bound?

Can any fixed number be an upper bound?

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$$S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 2$$

$$S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{8} > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) = 2\frac{1}{2}$$

Example (Harmonic Series) $\sum_{n=1}^{\infty} \frac{1}{n}$

Since each term is positive, seq. of partial sums (S_m)
 $S_m = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{m}$ is increasing.

Is (S_m) bounded?

Is 2 an upper bound? $S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) = 2$ ✓

Is 2.5 an upper bound? $S_8 = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{8} > 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \dots + \frac{1}{8}) = 2\frac{1}{2}$ ✓

Can any fixed number be an upper bound?

$$\begin{aligned}
 S_{2^k} &= 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \dots + \frac{1}{8}) + \dots + (\frac{1}{2^{k-1}+1} + \dots + \frac{1}{2^k}) \\
 &> 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \dots + \frac{1}{8}) + \dots + (\frac{1}{2^k} + \dots + \frac{1}{2^k}) \\
 &= 1 + \frac{1}{2} + 2(\frac{1}{4}) + 4(\frac{1}{8}) + \dots + 2^{k-1}(\frac{1}{2^k}) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2} = 1 + k(\frac{1}{2})
 \end{aligned}$$

Is 24 an upper bd.?

Pick $k=48$,

$$S_{2^{48}} > 1 + 48 \left(\frac{1}{2}\right) = 25$$

So, ~~24~~ cannot be an u. bound. for (S_n) .

This argument using $S_{2^k} > 1 + k \left(\frac{1}{2}\right)$

show (S_n) is unbdd.

$\Rightarrow (S_n)$ is not convergent $\left(\begin{array}{l} \because \text{convergent} \\ \Rightarrow \text{bdd.} \end{array} \right)$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent.

The Harmonic Series argument is a special instance of the following:

Theorem [Cauchy Condensation Test]

Suppose (b_n) is decreasing and $b_n \geq 0 \forall n$. Then,
$$\sum_{n=1}^{\infty} b_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n b_{2^n} \text{ converges}$$

Proof \Rightarrow Prove the contrapositive as we did with Harmonic Series

$$\sum_{n=0}^{\infty} 2^n b_{2^n} \text{ diverges} \Rightarrow \sum_{n=1}^{\infty} b_n \text{ diverges}$$

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$$\sum_{n=1}^{\infty} b_n \text{ converges} \iff \sum_{n=0}^{\infty} 2^n b_{2^n} \text{ converges}$$

Proof \Leftarrow Assume $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges

Then the seq. of partial sums (t_k) , $t_k = b_1 + 2b_2 + 4b_4 + \dots + 2^k b_{2^k}$ converges, and hence is bounded: $\exists M > 0$ s.t. $t_k \leq M \forall k$.

To show $\sum_{n=1}^{\infty} b_n$ converges, we only need to show $s_m = b_1 + b_2 + \dots + b_m$ is a bounded seq. (\because we already know (s_m) is increasing).

Fix m & pick k s.t. $m \leq 2^{k+1} - 1$. Then $s_m \leq s_{2^{k+1} - 1}$ (\leftarrow)

$$s_{2^{k+1} - 1} = b_1 + (b_2 + b_3) + (b_4 + \dots + b_7) + \dots + (b_{2^k} + \dots + b_{2^{k+1} - 1}) \leq ?$$

$$\begin{aligned}
S_{2^{k+1}-1} &= b_1 + (b_2 + b_3) + (b_4 + b_5 + b_6 + b_7) + \dots + (b_{2^k} + \dots + b_{2^{k+1}-1}) \\
&\rightarrow \leq b_1 + (b_2 + b_2) + (b_4 + \dots + b_4) + \dots + (b_{2^k} + \dots + b_{2^k}) \\
&= b_1 + 2b_2 + 4b_4 + \dots + 2^k b_{2^k} \\
&= t_k
\end{aligned}$$

(Why?)
since (b_n) dec.
 $b_1 \geq b_2 \geq b_3 \geq \dots$

Since $S_m \leq S_{2^{k+1}-1} \leq t_k \leq M \Rightarrow S_m \leq M \quad \forall m$

we get (S_m) is bounded.

\therefore by MCT, (S_m) converges. So, $\sum_{n=1}^{\infty} b_n$ converges.

Cor $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

~~Proof~~ Think!

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We showed seq. of partial sums (s_m) of the Harmonic Series does not converge by analyzing one particular sequence within (s_m) namely, $(S_{2^k}) = (s_1, s_2, s_4, s_8, s_{16}, \dots)$ \uparrow subsequence

Defn Let (a_n) be a sequence of real numbers.
Let $n_1 < n_2 < n_3 < \dots < \dots$ be an increasing sequence of natural numbers.

Then the sequence $(a_{n_1}, a_{n_2}, a_{n_3}, \dots)$ is called a subsequence of (a_n) , and is denoted by (a_{n_k}) , $k \in \mathbb{N}$.

$$(a_1, a_2, a_3, \dots)$$

$$n_1 < n_2 < n_3 < n_4 < \dots$$

$$(a_{n_1}, a_{n_2}, a_{n_3}, \dots)$$

The order of appearance does not change.

And no terms are repeated.

e.g. $(a_n) = (1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \dots)$

Which of these is a subsequence of (a_n) ?

① $(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots)$ Yes

② $(\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}, \frac{1}{10000}, \dots)$ Yes

③ $(\frac{1}{10}, \frac{1}{5}, \frac{1}{100}, \frac{1}{50}, \frac{1}{1000}, \dots)$ No

④ $(1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots)$

No

$\frac{1}{5}$ before $\frac{1}{10}$ $\frac{1}{50}$ before $\frac{1}{100}$

Theorem If a sequence (a_n) converges to a then every subsequence of (a_n) also converges to a .

Proof

Let (a_{n_k}) be any subseq. of (a_n) .

Given $\epsilon > 0$, $\exists N$ s.t. $|a_n - a| < \epsilon \ \forall n \geq N$. (since $(a_n) \rightarrow a$)

Since $1 \leq n_1 < n_2 < n_3 < \dots$, is an inc. seq. of natural numbers, $k \leq n_k$ for all k .

$\therefore |a_{n_k} - a| < \epsilon \ \forall k \geq N$ (\because $k \geq N \Rightarrow n_k \geq N \Rightarrow |a_{n_k} - a| < \epsilon$)

i.e., $(a_{n_k}) \rightarrow a$ as $k \rightarrow \infty$.

since
 $n_k \geq k \geq N$

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$\therefore |a_{n_k} - a| < \epsilon \quad \forall k \geq N$ ($\because k \geq N \Rightarrow n_k \geq N \Rightarrow |a_{n_k} - a| < \epsilon$)
for all k .

i.e., $(a_{n_k}) \rightarrow a$ as $k \rightarrow \infty$.

Cor Sequence (a_n) converges to $a \iff$ every subsequence of (a_n) converges to a

Divergence Criterion If a sequence contains two subsequences that converge to different limits, then the sequence cannot be convergent

e.g. Consider the sequence $(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, -\frac{1}{5}, \frac{1}{5}, \dots)$

$(\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \dots)$ is a subsequence that converges to $\frac{1}{5}$

$(-\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, -\frac{1}{5}, \dots)$ is a subsequence that converges to $-\frac{1}{5}$

\therefore The original sequence is not convergent

example What is the limit of (b^n) where $0 < b < 1$?

$$b^1 > b^2 > b^3 > \dots > 0$$

$\Rightarrow (b^n)$ is decreasing & bounded below.

By MCT, $(b^n) \rightarrow l$ for some $l \in \underline{[0, b]} \subseteq [0, 1)$

How to find l ?

(b^{2n}) is a subsequence. $\therefore (b^{2n}) \rightarrow \underline{l}$ (By Previous Thm.)

Also, $b^{2n} = b^n b^n$, product of two seq's. $\therefore (b^{2n}) \rightarrow \underline{l \cdot l = l^2}$ (By Algebra of limits)

\therefore $\textcircled{1} = \textcircled{2}$ gives $\underline{l^2 = l \Leftrightarrow l^2 - l = 0 \Leftrightarrow l(l-1) = 0} \Rightarrow \underline{l = 0}$
by uniqueness of limits

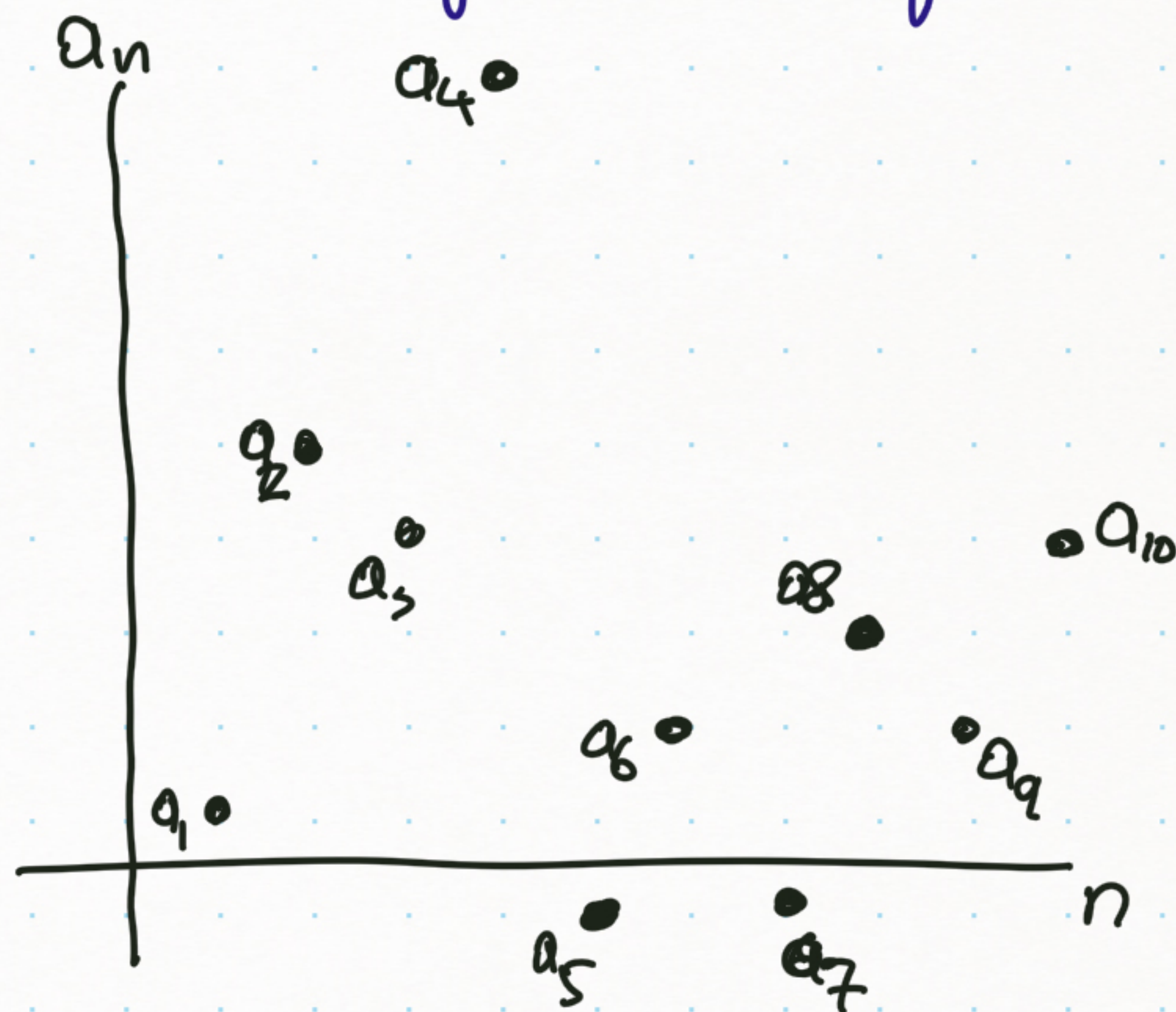
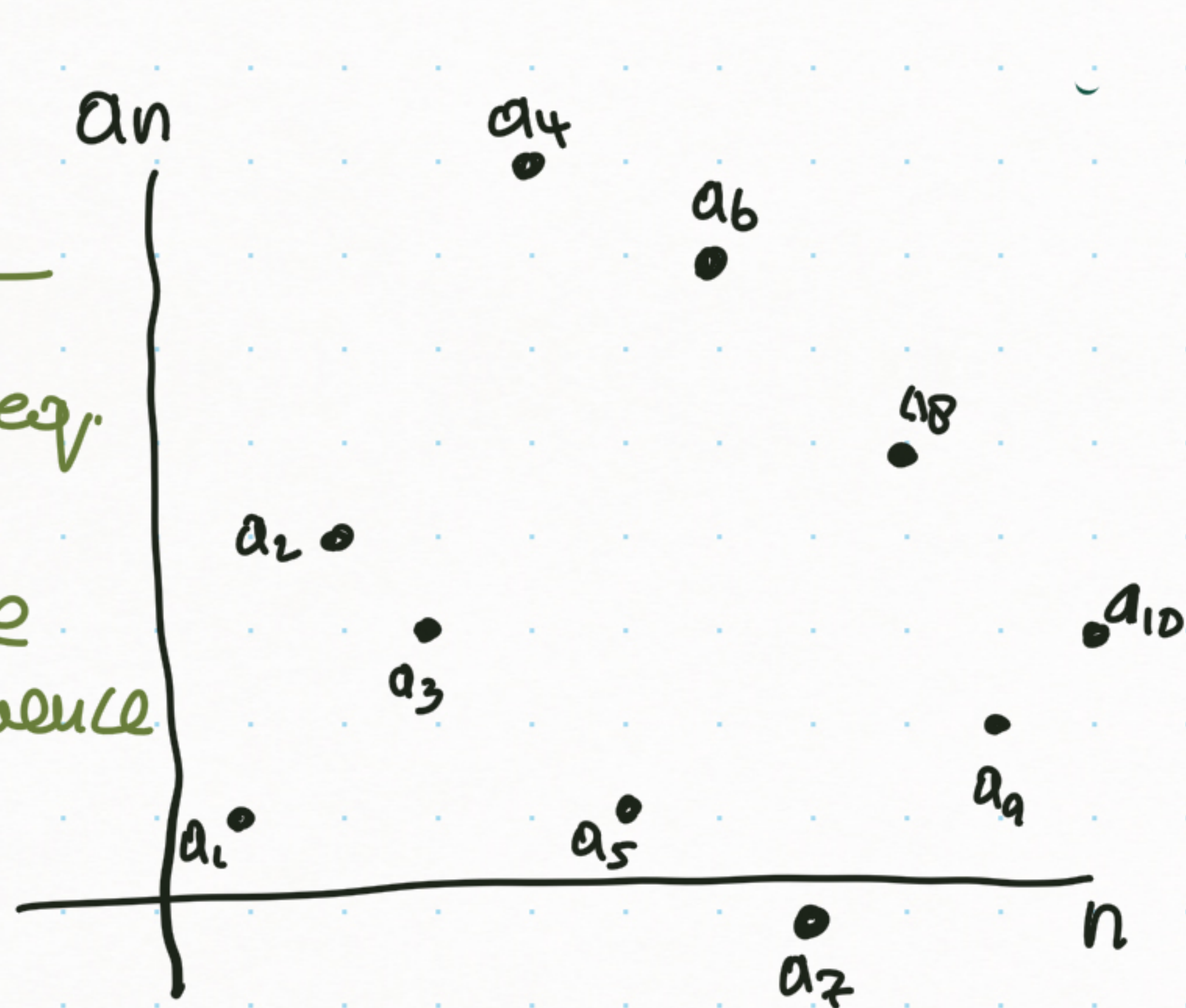
Can we always find a convergent subsequence even if the sequence is not convergent?

Bolzano - Weierstrass Theorem

Every bounded sequence contains a convergent subsequence.

Idea

Every seq. has a monotone subsequence



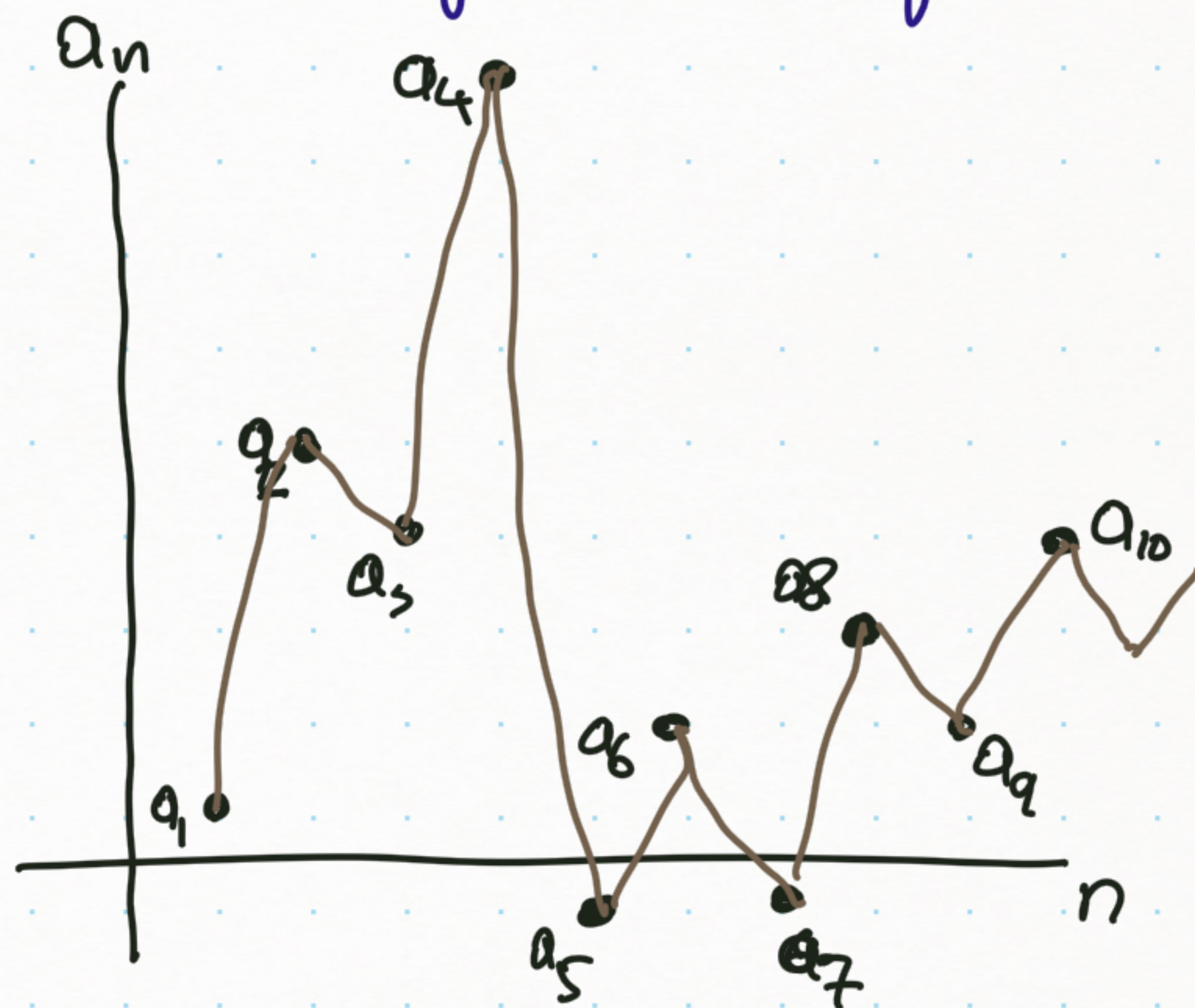
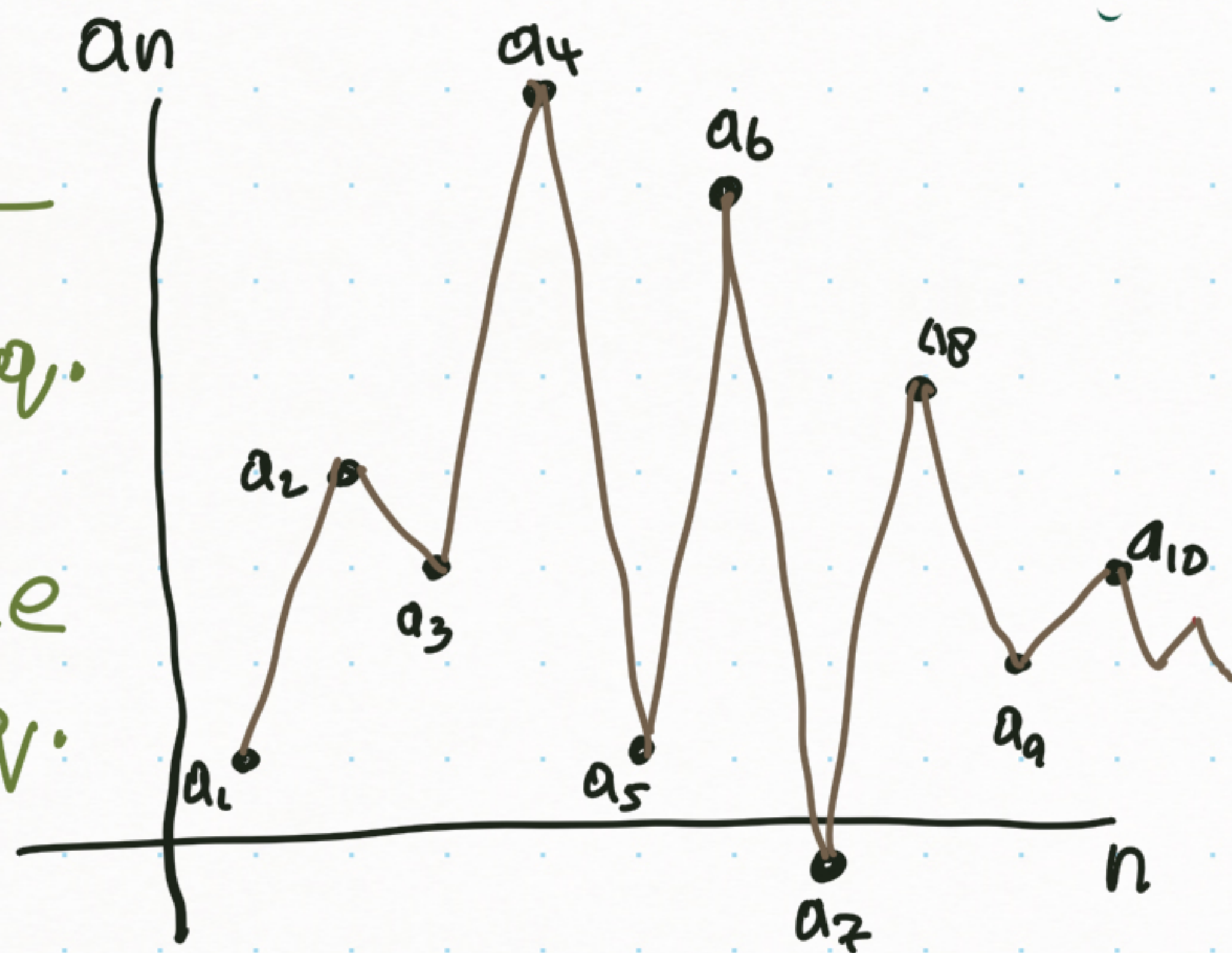
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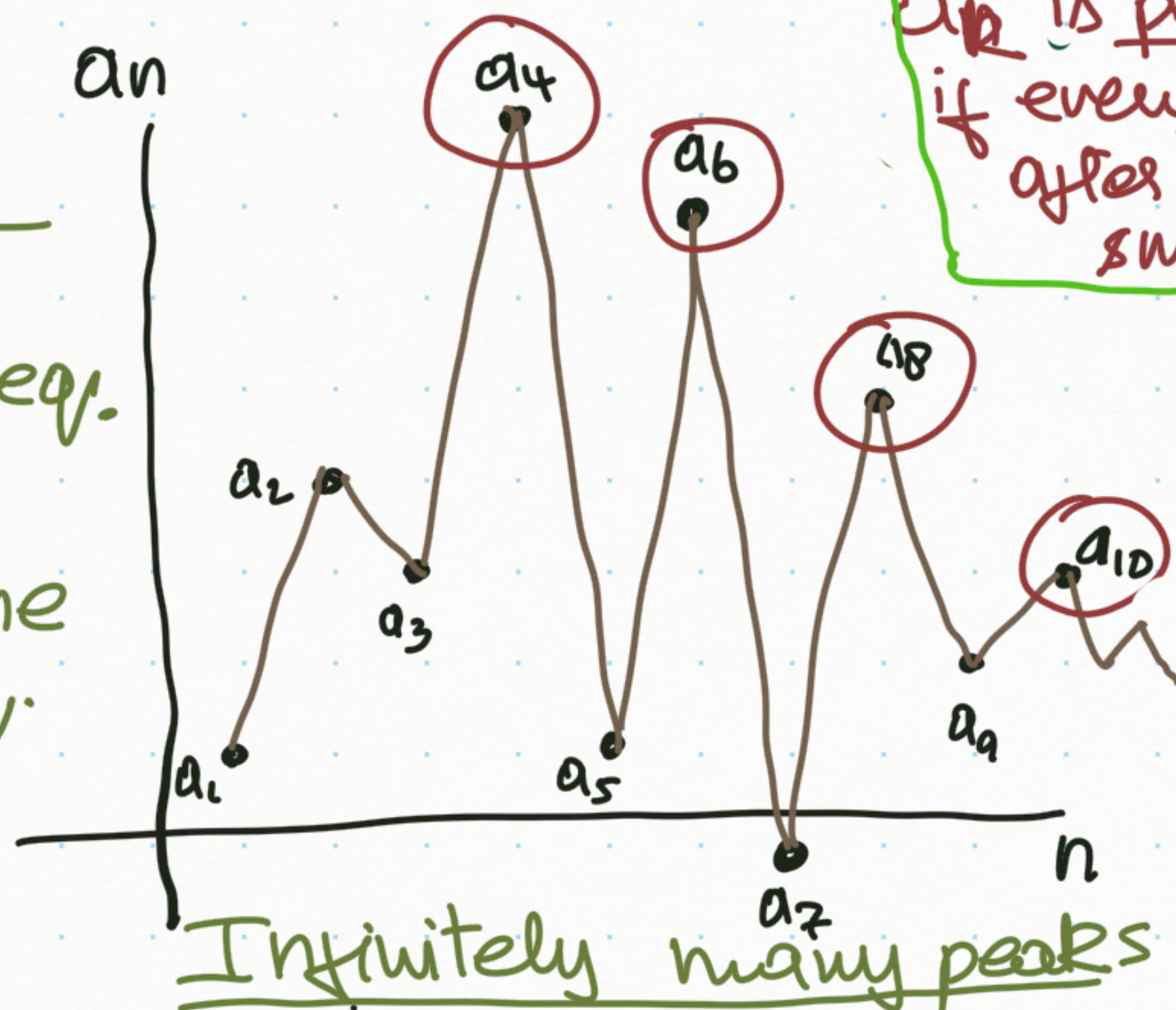


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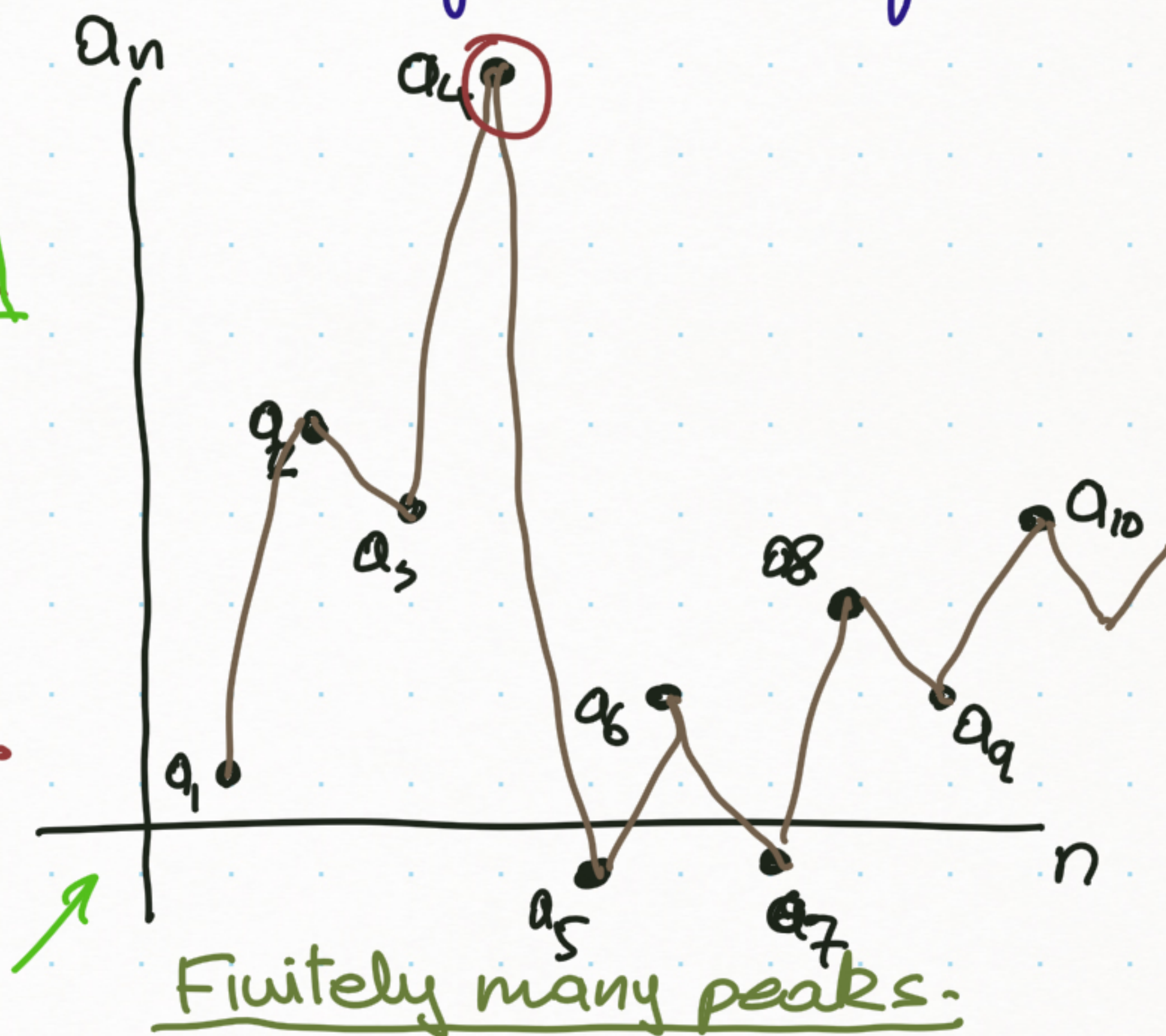
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Every seq. has a monotone subseq.



a_k is peak if every a_m after it is smaller.

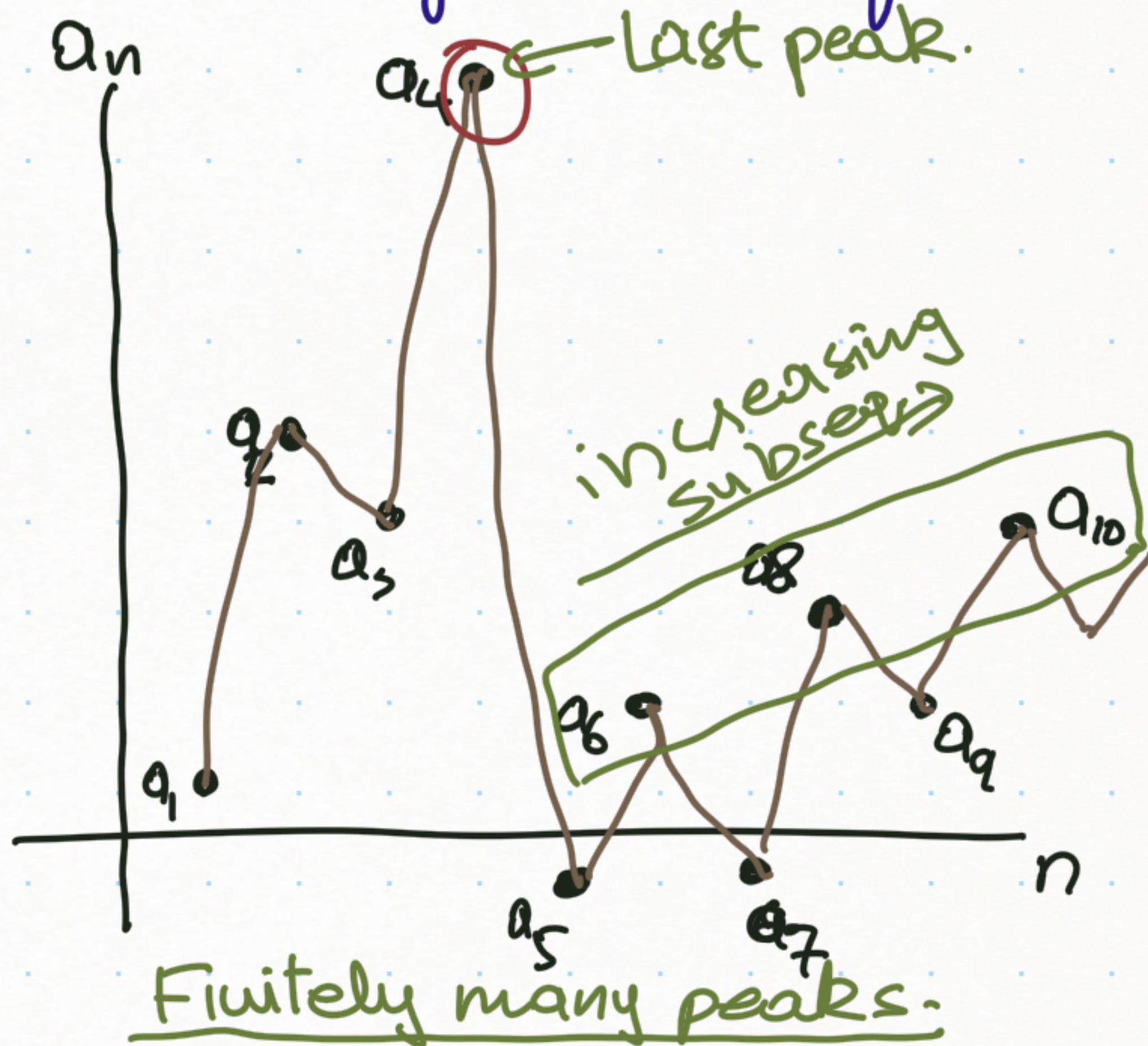
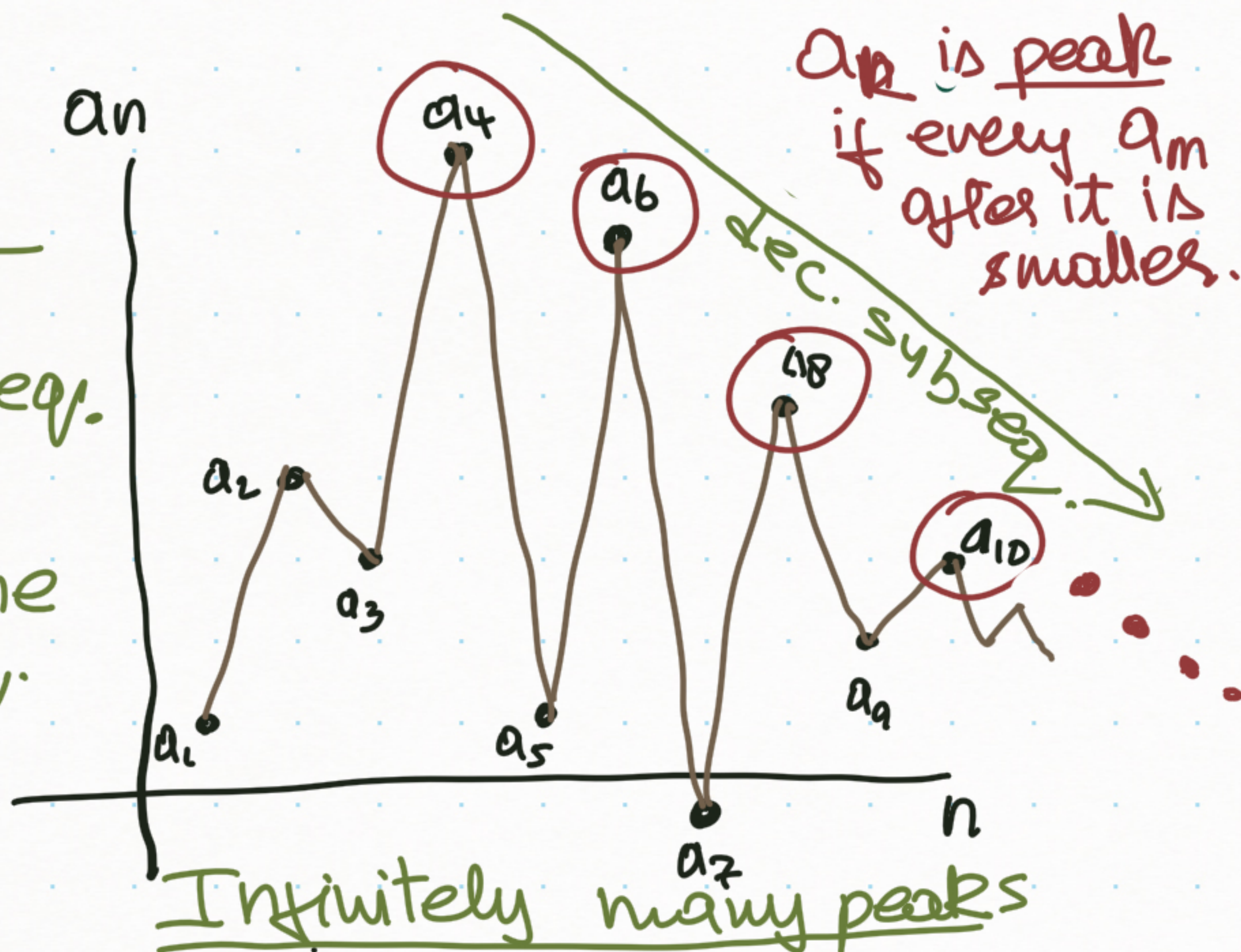


Can we always find a convergent subsequence even if the sequence is not convergent?

Bolzano - Weierstrass Theorem Every bounded sequence contains a convergent subsequence.

Idea

Every seq. has a monotone subseq.



Lemma Every sequence has a monotone subsequence.

Proof Call a_n a peak if $a_n \geq a_m \quad \forall m \geq n$.

Two possibilities \rightarrow a_n is at least as large as every term after it

Case 1 Infinitely many peaks

By defn. of peak, $a_{n_k} \geq a_{n_{k+1}} \quad \forall k$

Let a_{n_k} be the k^{th} peak

$\therefore (a_{n_k})$ is a dec. subseq.

Case 2 Finitely many peaks

Lemma Every sequence has a monotone subsequence.

Proof Call a_n a peak if $a_n \geq a_m \forall m \geq n$.

Two possibilities \rightarrow

Case 1 Infinitely many peaks Let a_{n_k} be the k^{th} peak.

By defn., of peak $a_{n_k} \geq a_{n_{k+1}} \forall k \quad \therefore (a_{n_k})$ is a dec. subseq.

Case 2 Finitely many peaks Let a_N be the last peak.

Let $a_{n_1} = a_{N+1}$. Since a_N was the last peak, a_{N+1} is not a peak & hence $\exists a_{n_2}$ after it that's larger than it. Then $a_{n_2} > a_{n_1}$

Now a_{n_2} is also not a peak. Hence, there is $a_{n_3} > a_{n_2}$ with $n_3 > n_2$.
This process gives us $a_{n_1} < a_{n_2} < a_{n_3} < \dots$, where $n_1 < n_2 < n_3 < \dots$.
Inc. subseq.

Proof of B-W Theorem [(a_n) bdd. $\Rightarrow \exists$ subseq. convergent]

Suppose (a_n) is a bdd. seq.

By Lemma, (a_n) has a monotone subseq. (a_{n_k}) .

And (a_{n_k}) is bounded (since (a_n) is bounded)

By MCT, (a_{n_k}) is convergent.

\therefore we found a convergent subseq. of (a_n) .



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• Definition of convergence requires knowledge of the limit of the sequence.

$$|a_n - \underline{a}| < \epsilon$$

• MCT is useful because it allows us to show convergence of a sequence without knowing its limit.

But MCT is only a sufficient condition for convergence.

→ Can we characterize convergence without knowing the limit?

- Definition of convergence requires knowledge of the limit of the sequence.
- MCT is useful because it allows us to show convergence of a sequence without knowing its limit.
But MCT is only a sufficient condition for convergence.

→ Can we characterize convergence without knowing the limit?

Maybe it means "each term is getting closer to the previous term"

i.e. $|a_{n+1} - a_n| < \epsilon \quad \forall n > N.$

example $(a_n) = (\sqrt{n})$

$$\begin{aligned} \underline{|a_{n+1} - a_n|} &= |\sqrt{n+1} - \sqrt{n}| = \sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} \iff \frac{1}{\sqrt{n}} < \epsilon \iff n > \frac{1}{\epsilon^2} \end{aligned}$$

Given $\epsilon > 0$, let $N = \frac{1}{\epsilon^2}$ Then $|a_{n+1} - a_n| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{\frac{1}{\epsilon^2}}} = \epsilon$
for all $n > N$.

$$\underline{|a_{n+1} - a_n| < \epsilon \quad \forall n > N}$$

example $(a_n) = (\sqrt{n})$

$$\begin{aligned} |a_{n+1} - a_n| &= |\sqrt{n+1} - \sqrt{n}| = \sqrt{n+1} - \sqrt{n} = (\sqrt{n+1} - \sqrt{n}) \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} \end{aligned}$$

Given $\epsilon > 0$, let $N = \frac{1}{\epsilon^2}$. Then $|a_{n+1} - a_n| < \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{\frac{1}{\epsilon^2}}} = \epsilon$
for all $n > N$.

(a_n) diverges even though each $|a_{n+1} - a_n|$ is small.

How to fix this?

Defn A sequence (a_n) is called Cauchy sequence
if $\forall \epsilon > 0 \exists N$ s.t. $|a_m - a_n| < \epsilon \forall m, n \geq N$.

We want all terms to be close to each other
not just consecutive terms.

Defn A sequence (a_n) is called Cauchy sequence if $\forall \epsilon > 0 \exists N$ s.t. $|a_m - a_n| < \epsilon$ $\forall m, n \geq N$.

e.g. $(7 + \frac{1}{n})$ is Cauchy.

$$|a_m - a_n| = |(7 + \frac{1}{m}) - (7 + \frac{1}{n})| = |\frac{1}{m} - \frac{1}{n}| \leq |\frac{1}{m}| + |\frac{1}{n}| \quad (\text{by } \Delta \text{ ineq.})$$
$$= |\frac{1}{m}| + |\frac{1}{n}| = \frac{1}{m} + \frac{1}{n}$$

Given $\epsilon > 0$, since $\epsilon/2 > 0$ \exists $N > 0$ s.t. $\frac{1}{N} < \frac{\epsilon}{2}$ (by Archimedean Principle)

For any $n, m > \underline{N}$,

$$|a_m - a_n| \leq |\frac{1}{m}| + |\frac{1}{n}| < \frac{1}{N} + \frac{1}{N}$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon$$

Lemma If (a_n) is Cauchy then (a_n) is bounded

Proof The Proof is essentially a repetition of the proof we did earlier for "If (a_n) is convergent then (a_n) is bounded".

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Proof The Proof is essentially a repetition of the proof we did earlier for "If (a_n) is convergent then (a_n) is bounded".

Given $\epsilon = 1$, $\exists N$ s.t. $|a_m - a_n| < 1 \quad \forall m, n \geq N$. Take $m = N$

so in particular, we have $|a_n - a_N| < 1 \quad \forall n \geq N$

$$\text{i.e., } a_N - 1 < a_n < a_N + 1 \quad \forall n \geq N$$

$$\text{i.e., } |a_n| < |a_N| + 1 \quad \forall n \geq N$$

Set $M = \max \{ |a_1|, |a_2|, |a_3|, \dots, |a_{N-1}|, |a_N| + 1 \}$

Then $|a_n| \leq M \quad \forall n$.

Theorem [Cauchy Criterion]

A sequence converges if and only if it is Cauchy.

Proof \Rightarrow Standard argument left as exercise.

Hint: To show $|a_m - a_n| < \epsilon$

write $|a_m - a_n| = |a_m - a + a - a_n|$ (where $\lim a_n = a$)

$$\begin{aligned} &\leq \quad ? \\ &< \quad ? \\ &= \quad \epsilon \end{aligned}$$

Theorem [Cauchy Criterion]

A sequence converges if and only if it is Cauchy.

Proof $\boxed{\Leftarrow}$

Idea

(a_n) Cauchy & hence bounded
Apply Bolzano-Weierstrass to get

$(a_{n_k}) \rightarrow a$ cont. subsequence

Then show $(a_n) \rightarrow a$ also.
using defn. of Cauchy.

Theorem [Cauchy Criterion]

A sequence converges if and only if it is Cauchy.

Proof $\boxed{\Leftarrow}$ Let (a_n) be Cauchy. By Lemma, (a_n) is bounded.

B-W \Rightarrow \exists a convergent subsequence (a_{n_k}) .

Let $a = \lim a_{n_k}$

Given $\epsilon > 0$, since (a_n) is Cauchy, $\exists N_1$ s.t. $|a_n - a_m| < \frac{\epsilon}{2} \forall n, m \geq N_1$ — ①

Since $(a_{n_k}) \rightarrow a$, $\exists N_2$ s.t. $|a_{n_k} - a| < \frac{\epsilon}{2} \forall n_k \geq N_2$. — ②

Set $N = \max\{N_1, N_2\}$. Then for any $m \geq N$,

$$\underline{|a_m - a|} = \dots < \epsilon$$

Theorem [Cauchy Criterion]

A sequence converges if and only if it is Cauchy.

Proof $\boxed{\Leftarrow}$ Let (a_n) be Cauchy. By Lemma, (a_n) is bounded.

B-W $\Rightarrow \exists$ a convergent subsequence (a_{n_k}) .

Let $a = \lim a_{n_k}$

Given $\epsilon > 0$, since (a_n) is Cauchy, $\exists N_1$ s.t. $|a_n - a_m| < \frac{\epsilon}{2} \forall n, m \geq N_1$ $\text{---} \textcircled{1}$

Since $(a_{n_k}) \rightarrow a$, $\exists N_2$ s.t. $|a_{n_k} - a| < \frac{\epsilon}{2} \forall n_k \geq N_2$. $\text{---} \textcircled{2}$

Set $N = \max\{N_1, N_2\}$. Then for any $m \geq N$,

$$|a_m - a| = |a_m - a_{n_N} + a_{n_N} - a| \leq |a_m - a_{n_N}| + |a_{n_N} - a|$$

by $\textcircled{1}$ since $n_N \geq N \geq N_1$ $\text{---} \textcircled{1}$
den. of subseq. $\text{---} \textcircled{1}$

$\text{---} \textcircled{2}$ By $\textcircled{2}$ since $n_N \geq N \geq N_2$

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Recall For an infinite series $\sum_{k=1}^{\infty} a_k$

- the sequence of terms is (a_1, a_2, a_3, \dots)
- the sequence of partial sums is (s_1, s_2, s_3, \dots)
where $s_m = a_1 + a_2 + a_3 + \dots + a_m$.

• $\sum_{k=1}^{\infty} a_k$ is convergent $\Leftrightarrow (s_m)$ is convergent

• If convergent, $\sum_{k=1}^{\infty} a_k = A \Leftrightarrow \lim_{m \rightarrow \infty} s_m = A$

Theorem [Algebra of Series Limits]

g₁ $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

(i) $\sum_{k=1}^{\infty} (ca_k) = cA$ for all $c \in \mathbb{R}$,

(ii) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$

Proof Simple application of Algebra of Sequence limits.

$$\begin{aligned} \sum_{k=1}^{\infty} (a_k + b_k) &\stackrel{\textcircled{1}}{=} \lim_{m \rightarrow \infty} \left(\sum_{k=1}^m (a_k + b_k) \right) \stackrel{\textcircled{2}}{=} \lim_{m \rightarrow \infty} \left(\left(\sum_{k=1}^m a_k \right) + \left(\sum_{k=1}^m b_k \right) \right) \\ &\stackrel{\textcircled{3}}{=} \lim_{m \rightarrow \infty} \left(\sum_{k=1}^m a_k \right) + \lim_{m \rightarrow \infty} \left(\sum_{k=1}^m b_k \right) \stackrel{\textcircled{4}}{=} A + B \end{aligned}$$

What are the reasons for each step $\textcircled{1}, \textcircled{2}, \textcircled{3}, \textcircled{4}$?

Theorem [Cauchy Criterion for Series]

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \forall \epsilon > 0, \exists N \text{ s.t.}$$

$|a_{m+1} + a_{m+2} + \dots + a_n| < \epsilon$

 $\forall n > m \geq N$

Proof $|S_n - S_m| = |a_{m+1} + a_{m+2} + \dots + a_n|$ ← Apply Cauchy criterion for seq.

$$\sum a_k \text{ convergent} \iff (S_m) \text{ is convergent}$$
$$\iff (S_m) \text{ is a Cauchy seq.}$$
$$\iff \forall \epsilon > 0 \exists N \text{ s.t. } \forall n, m \geq N, |S_m - S_n| < \epsilon$$

Theorem [Cauchy Criterion for Series]

$$\sum_{k=1}^{\infty} a_k \text{ converges} \iff \forall \epsilon > 0, \exists N \text{ s.t.}$$
$$\underline{|a_{m+1} + a_{m+2} + \dots + a_n| < \epsilon \quad \forall n > m \geq N}$$
$$|a_n| < \epsilon \text{ i.e., } |a_n - 0| < \epsilon$$

Cor If $\sum_{k=1}^{\infty} a_k$ converges then $(a_k) \rightarrow 0$

Proof Apply Cauchy criterion for series with $n = m+1$.

Theorem [Comparison Test]

Suppose the sequences (a_n) & (b_n) satisfy $0 \leq a_k \leq b_k \forall k$

(i) If $\sum_{k=1}^{\infty} b_k$ converges then

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges then

$$\sum \frac{1}{\log n + 1} \gg \sum \frac{1}{n} \gg \sum \frac{1}{n^2 - n} \gg \sum \frac{1}{n^2} \gg \sum \frac{1}{n^2 + n}$$

?

↑
Dgt.

?

↑
Cgt.

?

Recall →

[Series p-test] $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\Leftrightarrow p > 1$.

Proof ① If $p \leq 1$ then $\frac{1}{n} \leq \frac{1}{n^p}$ $\forall n$

Use this to show $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent

(Using Comp. T. with $\sum \frac{1}{n}$ div.)

② If $p > 1$ then use Cauchy Condensation Test.

[Geometric Series] A Geometric series is of the form:

$$\sum_{k=0}^{\infty} ar^k = \underbrace{a + ar + ar^2 + \dots}_{\text{is convergent} \Leftrightarrow |r| < 1} \quad \text{for some fixed } a \text{ \& } r. \quad \text{Then the sum} = \frac{a}{1-r}$$

If $a=0$ then sum = 0

If $a \neq 0$ and $r=1$ then divergent

If $a \neq 0$ and $r \neq 1$ then

$$S_m = \underbrace{a + ar + \dots + ar^{m-1}}_{\text{seq. of partial sums}} = \frac{a(1-r^m)}{1-r}$$

using the identity

$$(1-r)(1+r+r^2+\dots+r^{m-1}) = 1-r^m$$

$$\lim_{m \rightarrow \infty} S_m = \lim_{m \rightarrow \infty} \frac{a(1-r^m)}{1-r}$$

exists $\Leftrightarrow |r| < 1$

$$= \left[\frac{a}{1-r} \right] \left(\lim_{m \rightarrow \infty} (1-r^m) \right) \quad \text{limit exists} \Leftrightarrow |r| < 1$$
$$= \left[\frac{a}{1-r} \right] \left(1 - \lim_{m \rightarrow \infty} r^m \right) = \frac{a}{1-r} \Leftrightarrow |r| < 1$$

Math 400

Real Analysis

Video # 17

What if our series contains negative terms also?

Theorem [Absolute Convergence Test]

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges as well.

What if our series contains negative terms also?

Theorem [Absolute Convergence Test]

If $\sum_{n=1}^{\infty} |a_n|$ converges then $\sum_{n=1}^{\infty} a_n$ converges as well.

Proof (Apply Cauchy criterion).

Given $\epsilon > 0$, since $\sum |a_n|$ converges, by Cauchy criterion $\exists N$ s.t. $|a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \epsilon \quad \forall n > m \geq N$.

By Δ -inequality,

$$|a_{m+1} + \dots + a_n| < |a_{m+1}| + \dots + |a_n| < \epsilon \quad \forall n > m \geq N$$

Hence by Cauchy criterion, $\sum a_n$ converges.

Theorem [Alternating Series Test]

Let (a_n) satisfy (i) $a_1 \geq a_2 \geq \dots \geq a_n \geq a_{n+1} \geq \dots$, and

(ii) $(a_n) \rightarrow 0$

$(a_n) \downarrow 0$

Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof [HW Exercise]

Proof #1 → Show (S_m) is a Cauchy seq.
 ← seq. of partial sums

Proof #2

→ Show subsequences (S_{2m}) and (S_{2m+1}) are both

convergent by MCT
 & that implies (S_m) is also convergent.

example $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ (alternating harmonic series)

is convergent by Alternating Series Test.

But $\sum_{k=1}^{\infty} \left| (-1)^{k+1} \frac{1}{k} \right| = \sum_{k=1}^{\infty} \frac{1}{k}$ (Harmonic series) is divergent.

Do you know the value of the limit?

Definition If $\sum_{k=1}^{\infty} |a_k|$ converges then we say $\sum_{k=1}^{\infty} a_k$ converges absolutely.

If $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges then we say $\sum_{k=1}^{\infty} a_k$ converges conditionally.

Examples for each type?

A rearrangement of a series is obtained by permuting the terms in the sum into some other order

- all the original terms eventually appear
- no original term is repeated
- no new terms are introduced.

Defn A rearrangement of $\sum_{k=1}^{\infty} a_k$ is a series $\sum_{k=1}^{\infty} b_k$ for which there is a bijection $f: \mathbb{N} \rightarrow \mathbb{N}$ s.t. $b_{f(k)} = a_k \forall k$.

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

$\frac{1}{5} + \frac{1}{3} - \frac{1}{2} - \frac{1}{6} + 1 + \frac{1}{10871} + \frac{1}{7} - \frac{1}{4} + \dots$ rearrangement

In the very first video we used rearrangement of series to show how that can lead to contradictions.

Look at $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

We know it's convergent & hence equals some number S .

Then, $\frac{1}{2}S = 0 - \frac{1}{2} + 0 - \frac{1}{4} + 0 - \frac{1}{6} + 0 - \frac{1}{8} + \frac{1}{10} \dots$
add to
 $S = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \dots$

to get
 $\frac{3}{2}S = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} \dots$

$\frac{3}{2}S = S$ ✗

Rearrangement of $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}$

Two positive terms followed by negative...

[Rearrangement Theorem I]

If $\sum_{k=1}^{\infty} a_k$ converges conditionally then

for any L ($L \in \mathbb{R}$ or $L = \pm\infty$)

\exists rearrangement of $\sum_{k=1}^{\infty} a_k$ that converges to L

[Rearrangement Theorem I]

If $\sum_{k=1}^{\infty} a_k$ converges conditionally, then

for any L ($L \in \mathbb{R}$ or $L = \pm\infty$)

\exists rearrangement of $\sum_{k=1}^{\infty} a_k$ that converges to L

[Rearrangement Theorem II]

If $\sum_{k=1}^{\infty} a_k$ converges absolutely, then

every rearrangement of this series converges to the same limit.

Idea for RT-I

Suppose $L \in \mathbb{R}^+$

Since $\sum a_n$ converges conditionally it must have both positive and negative terms.

Let p_k be the k^{th} positive term of (a_n)

n_k be the k^{th} negative term of (a_n)

• $\sum a_n$ converges $\Rightarrow \underline{(a_n) \rightarrow 0}$

• $\sum a_n$ converges conditionally then

$$\begin{aligned} \sum_{k=1}^{\infty} p_k &= \infty && \text{(while } (p_k) \rightarrow 0) \\ \sum_{k=1}^{\infty} n_k &= -\infty && \text{(while } (n_k) \rightarrow 0) \end{aligned}$$

$$\sum p_k = \infty \Rightarrow \exists P_1 \text{ s.t. } \sum_{k=1}^{P_1} p_k > L$$

$$\sum n_k = -\infty \Rightarrow \exists N_1 \text{ s.t. } \sum_{k=1}^{P_1} p_k + \sum_{k=1}^{N_1} n_k < L$$

Repeat

$$\sum_{k=1}^{P_1} p_k + \sum_{k=1}^{N_1} n_k + \sum_{k=1}^{P_2} p_k > L \rightarrow \sum_{k=1}^{P_2} p_k + \sum_{k=1}^{N_1} n_k + \sum_{k=1}^{P_2} p_k + \sum_{k=1}^{N_2} n_k < L \rightarrow \dots$$