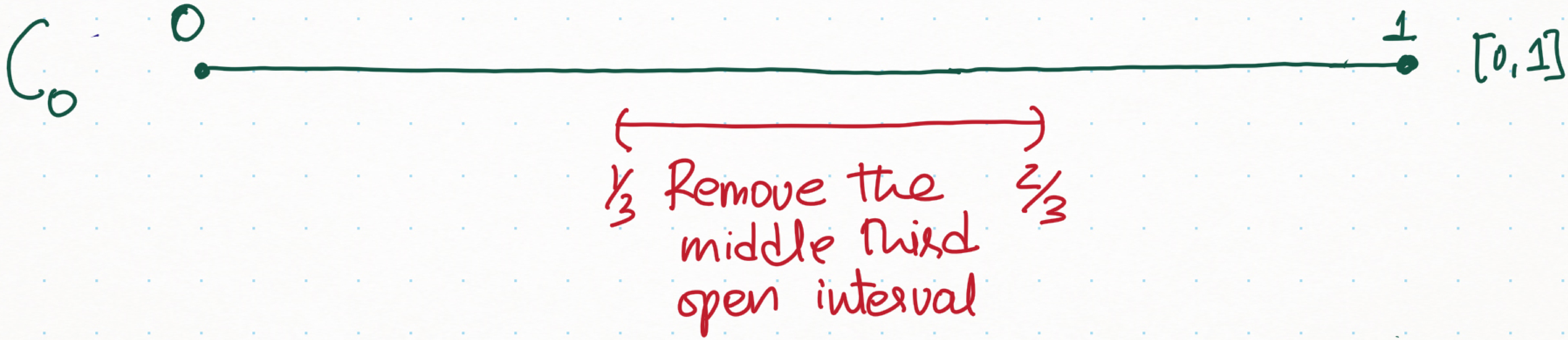
Math 400

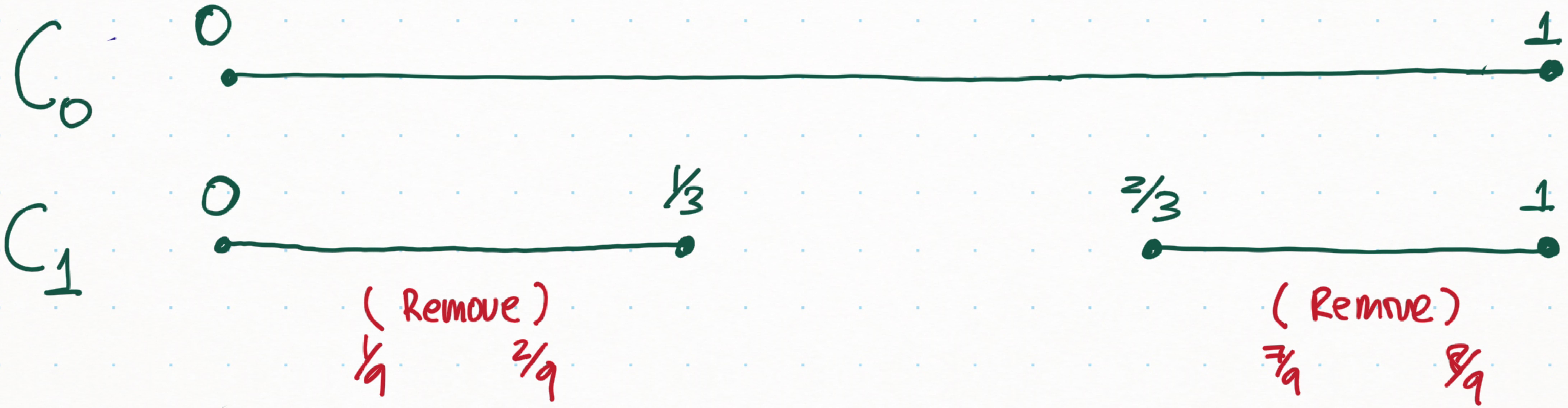
Real Analysis

Video #18

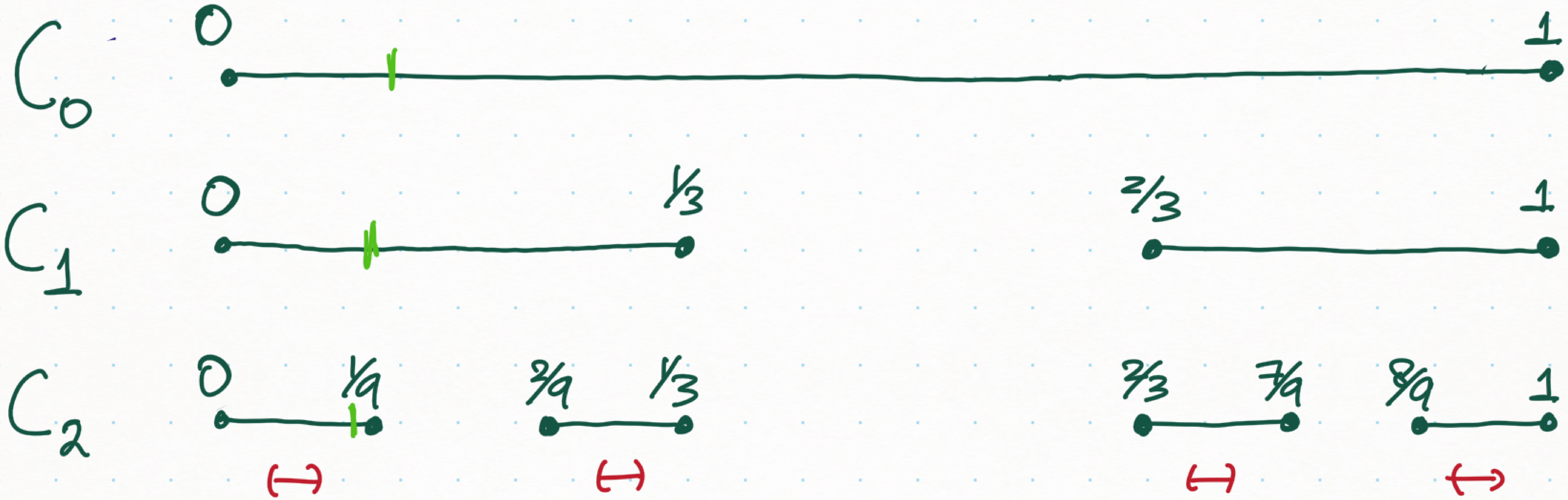
The Cantor Set



The Cantor Set



The Cantor Set



Keep removing the middle third open interval from each surviving interval.

⋮

C_n will consist of 2^n closed intervals each of length $1/3^n$ for $n = 0, 1, 2, \ldots\ldots$

Defn Cantor Set $C = \bigcap_{n=0}^{\infty} C_n$

In other words, $C = [0,1] \setminus \underbrace{\left[\left(\frac{1}{3},\frac{2}{3}\right) \cup \left(\frac{1}{9},\frac{2}{9}\right) \cup \left(\frac{7}{9},\frac{8}{9}\right) \cup \ldots\right]}_{\text{intervals that were removed}}$

Ques Is $C \neq \phi$?

- $C \neq \emptyset$ $\quad 0, 1 \in C_n \ \forall n$

In fact, endpoints of the closed intervals are never removed.

Ques How "big" is C?

• $C \neq \emptyset$ $\qquad 0, 1 \in C_n \ \forall n$

In fact, endpoints of the closed intervals are never removed.

Ques How "big" is C?

$\longrightarrow$ Cardinality

$\longrightarrow$ Length

$\longrightarrow$ Dimension

Cardinality: C is uncountable!

We can find a bijection between C and all binary sequences $\{(a_n)_{n=1}^{\infty} : a_n = 0 \text{ or } 1\}$

For $c \in C$, $a_1 = 0$ if c belongs to the left-hand interval of C_1

$a_1 = 1$ if ——— " ——— right-hand ——— " ———

Based on $a_1 \to$ $a_2 = 0$ if c belongs to the left-hand interval of the component of C_2 (as indicated by a_1)

$a_2 = 1$ if ——— " ——— right-hand ——— " ———

——— " ———

$\vdots$

Every point yields a sequence and every sequence defines a point.

Length C has length zero!

length of C = length of $[0,1]$

$-$ length of all the removed intervals

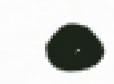
$$= 1 - \left(\frac{1}{3} + 2\left(\frac{1}{9}\right) + 4\left(\frac{1}{27}\right) + \cdots + 2^{n-1}\left(\frac{1}{3^n}\right) + \cdots\right)$$

$$= 1 - \left(\frac{1/3}{1-2/3}\right)$$

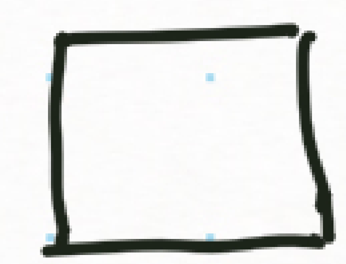
$$= 1 - 1$$

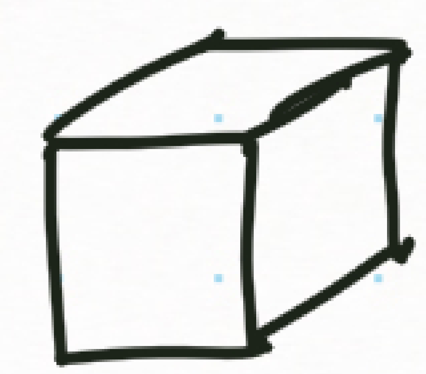
$$= 0$$

Dimension $\quad \dim(C) = 0.631\ldots$ (!!)

we all agree dimension of  is zero

 is one

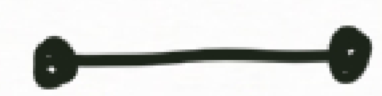
 is two

 is three

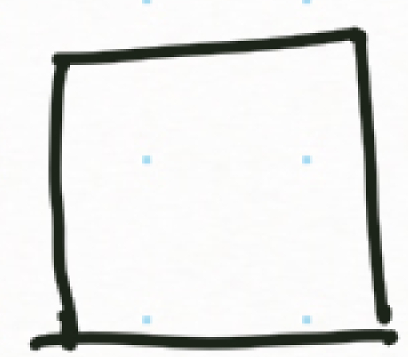
What happens when we magnify each such set by a factor of 3 ?

point

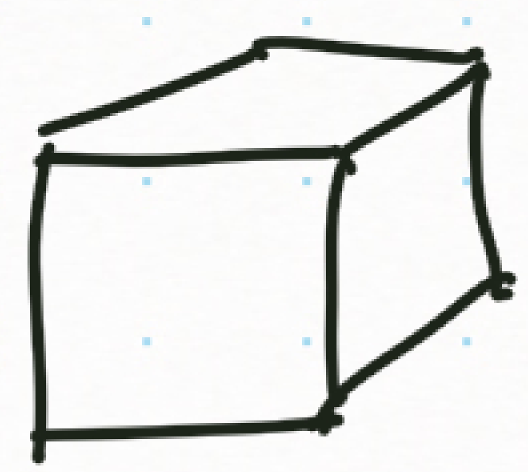
line segment



square



Cube

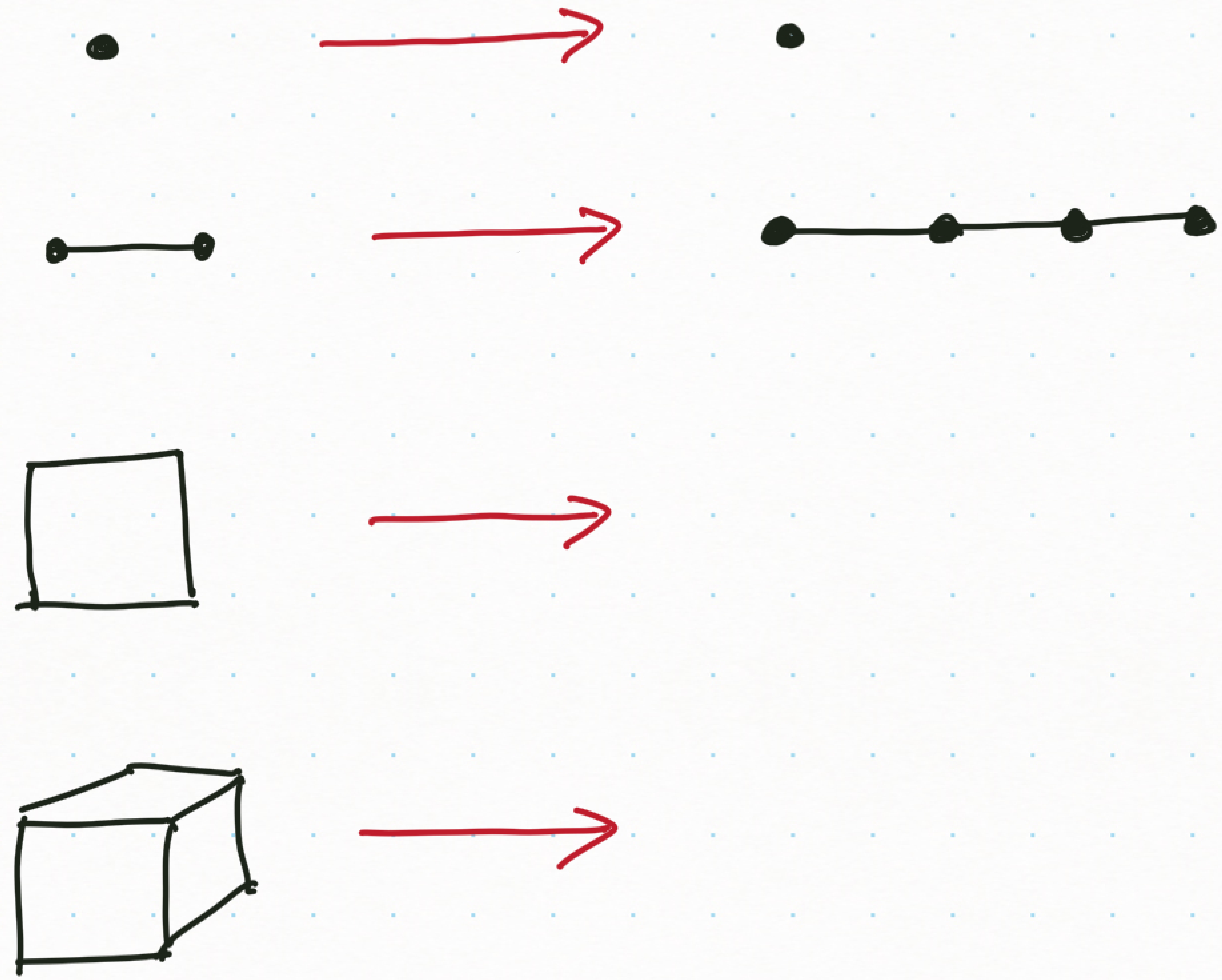


point

line segment

square

Cube

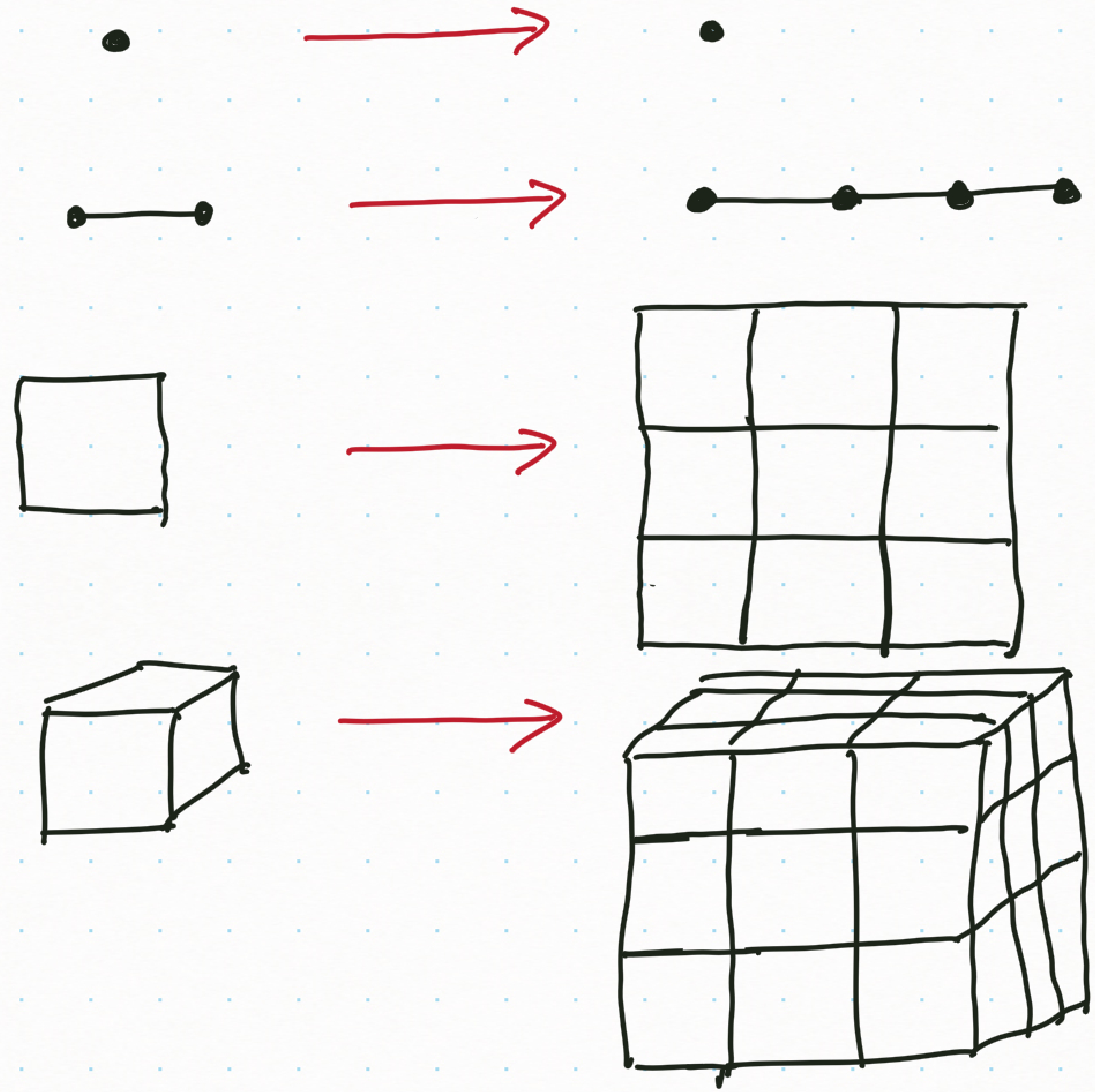


point

line segment

square

Cube

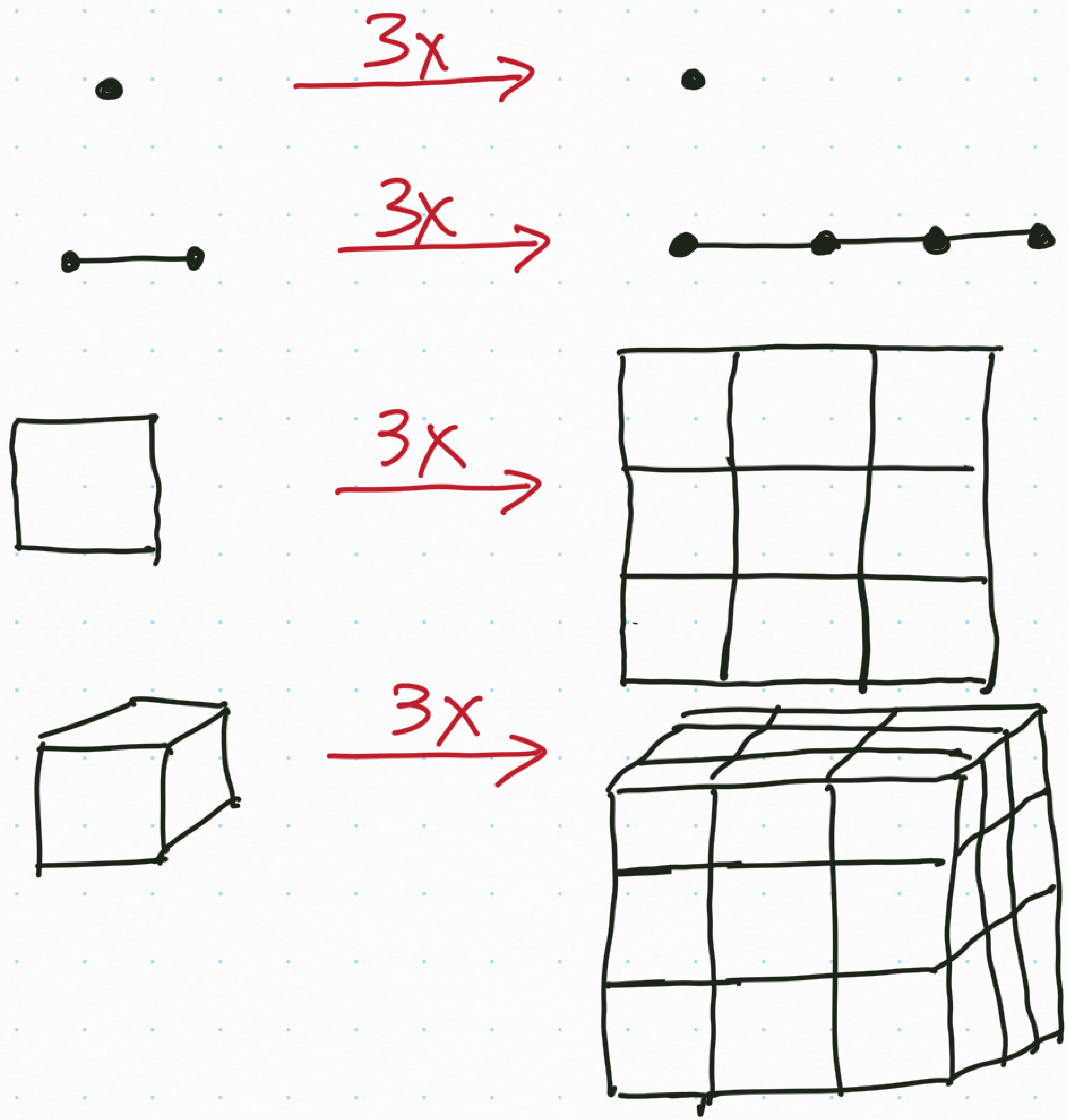


point — $1 = 3^0$ copies

line segment — $3 = 3^1$ copies

square — $9 = 3^2$ copies

Cube — $27 = 3^3$ copies



$3^{\text{dimension}}$

How many 3^d copies of C are created?

$C_0 = [0, 1]$ becomes $[0, 3]$

and C_1 becomes $[0, 1] \cup [2, 3]$

$$\underset{\text{copy of } C}{\underbrace{[0,1]}} \cup \underset{\text{copy of } C}{\underbrace{[2,3]}}$$

$\therefore C$ magnifies to $\underline{2}$ copies of C.

$$\underline{2 = 3^d} \iff d = \frac{\log 2}{\log 3} \simeq 0.631\ldots$$

Math 400

Real Analysis

Video #19

Recall For $a \in \mathbb{R}$ and $\epsilon > 0$
ϵ-neighborhood of a is $V_\epsilon(a) = \{x \in \mathbb{R} : |x-a| < \epsilon\}$
$= (a-\epsilon, a+\epsilon)$

Defn Set $O \subseteq \mathbb{R}$ is open if $\forall a \in O \ \exists \epsilon > 0$ s.t.
$V_\epsilon(a) \subseteq O$

Examples

Recall For $a \in \mathbb{R}$ and $\epsilon > 0$

ϵ-neighborhood of a is $V_\epsilon(a) = \{x \in \mathbb{R} : |x-a| < \epsilon\}$

$$= (a-\epsilon, a+\epsilon)$$

Defn Set $O \subseteq \mathbb{R}$ is open if $\forall a \in O \ \exists \epsilon > 0$ s.t.

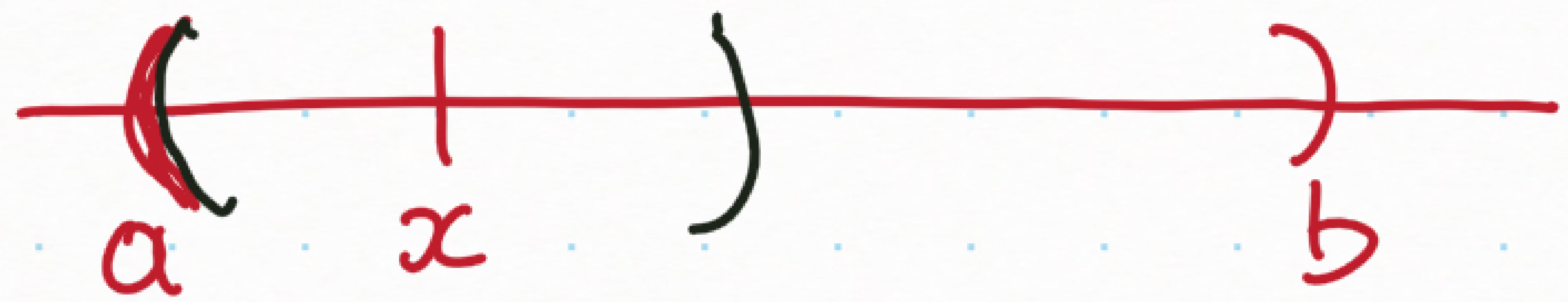
$$V_\epsilon(a) \subseteq O$$

Examples

- $\mathbb{R}$ is open. Let $a \in \mathbb{R}$. Then pick $\epsilon = 1$ & $V_1(a) = (a-1, a+1) \subseteq \mathbb{R}$
- $\emptyset$ is open

- Open interval (a,b) is open

Let $x \in (a,b)$ Pick $\epsilon = ?$ so that $V_\epsilon(x) = (x-\epsilon, x+\epsilon) \subseteq (a,b)$



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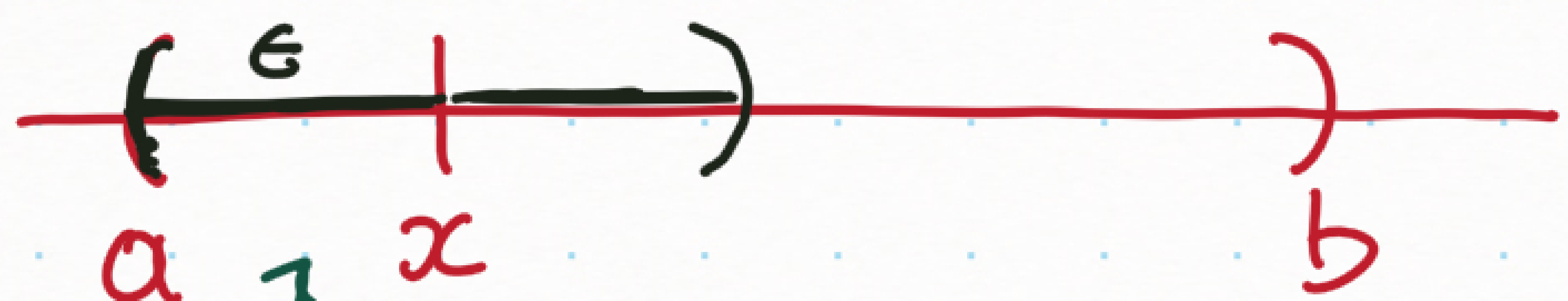


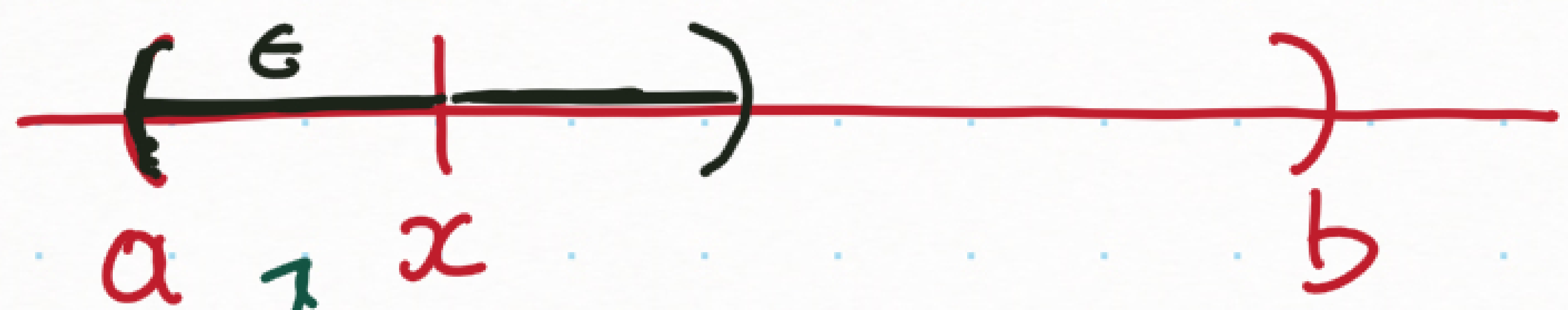
Pick $\epsilon = x-a$?

But what if

?

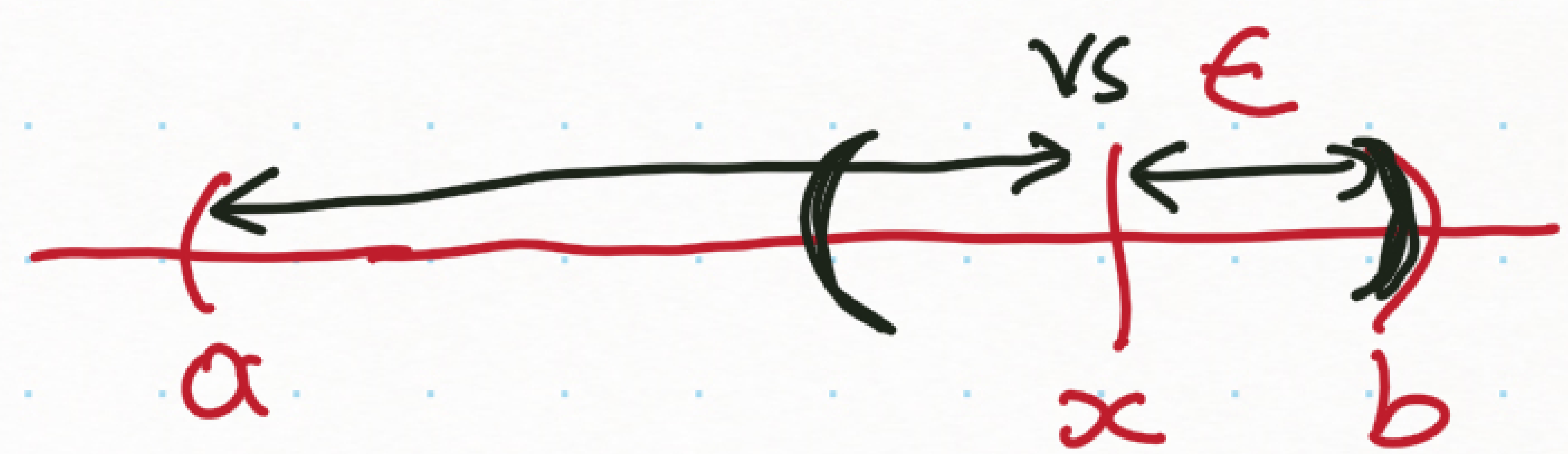
- <u>Open interval (a,b) is open</u>

Let $x \in (a,b)$ Pick $\epsilon = ?$ so that $V_\epsilon(x) = (x-\epsilon, x+\epsilon) \subseteq (a,b)$



Pick $\epsilon = x - a$?

But what if



?

Check!

$$\boxed{\begin{array}{l}\text{Pick } \underline{\epsilon = \min\{x-a,\, b-x\}} \\ \quad \text{then } V_\epsilon(x) = (x-\epsilon, x+\epsilon) \subseteq (a,b)\end{array}}$$

- Intervals (a,∞) and $(-\infty,b)$ are open

What ϵ should we use in each case?

- $[2,5]$ is not an open set

We have to argue: $\underbrace{\exists\, x \in [2,5] \text{ s.t. } V_\epsilon(x) = (x-\epsilon, x+\epsilon) \not\subseteq [2,5] \quad \forall \epsilon > 0.}_{\text{negation of defn. of open set}}$

What $x \in [2,5]$ should we use? $x=4$ ✗

Pick $x = 2$ Verify $(2-\epsilon, 2+\epsilon) \not\subseteq [2,5]$

Yes, $2-\epsilon < 2$

any index set
e.g. $\{1,2,\dots,k\}$, $\mathbb{N}$, $\mathbb{R}$, $\mathcal{P}(\mathbb{R})$, ...

Theorem ① If $\{O_\lambda : \lambda \in \Lambda\}$ is any collection of open sets then $\bigcup_{\lambda \in \Lambda} O_\lambda$ is also open.

② If $\{O_1, O_2, \dots O_k\}$ is a finite collection of open sets then $\bigcap_{i=1}^{k} O_i$ is also open.

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Since $a \in \bigcup_{\lambda \in \Lambda} O_\lambda$, $a \in O_{\lambda'}$ for some $\lambda' \in \Lambda$

$O_{\lambda'}$ is open, so $\exists\, \epsilon > 0$ s.t. $V_\epsilon(a) \subseteq O_{\lambda'}$ which is $\subseteq \bigcup_{\lambda \in \Lambda} O_\lambda$

as needed.

any index set
e.g. $\{1,2,\dots,k\}$, $\mathbb{N}$, $\mathbb{R}$, $\mathcal{P}(\mathbb{R})$, ...

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Proof ② Let $a \in \bigcap_{i=1}^{k} O_i$. Since $a \in O_i \ \forall i$ and each O_i is open, we have $\exists\, \epsilon_1 > 0, \epsilon_2 > 0, \dots, \epsilon_k > 0$ s.t. $V_{\epsilon_i}(a) \subseteq O_i \ \forall i = 1, \dots, k$.
But we need one $\epsilon > 0$ s.t. $V_\epsilon(a) \subseteq \bigcap_{i=1}^{k} O_i$ i.e., $V_\epsilon(a) \subseteq O_i \ \forall i = 1, \dots, k$.

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But we need one $\epsilon > 0$ s.t. $V_\epsilon(a) \subseteq \bigcap_{i=1}^{k} O_i$ i.e., $V_\epsilon(a) \subseteq O_i \ \forall i = 1, .., k$.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2, \dots, \epsilon_k\}$ then $V_\epsilon(a) \subseteq V_{\epsilon_i}(a) \ \forall i$, so, $V_\epsilon(a) \subseteq O_i \ \forall i$.

Apply the previous theorem to our examples of open sets:
open intervals of the form (a,b), $(-\infty,a)$, (b,∞).
What kind of sets do you get when you take unions or intersections of those open intervals?

Apply the previous theorem to our examples of open sets:
open intervals of the form (a,b), $(-\infty,a)$, (b,∞).

What kind of sets do you get when you take unions or intersections of those open intervals?

You will always get a union of open intervals.

e.g. $\big((-\infty,a)\cup(b,\infty)\big)\cap\big((-\infty,c)\cup(d,\infty)\big)$

$= (-\infty, \min\{a,c\}) \cup (\max\{b,d\},\infty)$

Theorem Every open set is a countable union of disjoint open intervals.

Proof (Outline)

Let $\mathcal{O}$ be an open set

Each $x \in \mathcal{O}$ is contained in $(x-\epsilon, x+\epsilon) \subseteq \mathcal{O}$ for some $\epsilon > 0$

Let I_x be the largest open interval in $\mathcal{O}$ that contains x

??

$I_x = (\alpha, \beta)$ What is α? β?

s.t. $x \in I_x$ ✓

and $I_x \subseteq \mathcal{O}$ ✓

Theorem Every open set is a countable union of disjoint open intervals.

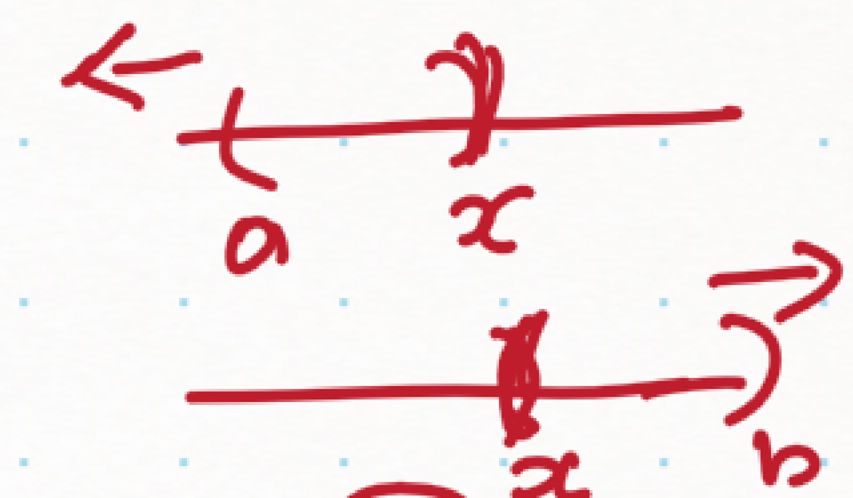
Proof (Outline)

Let $\mathcal{O}$ be an open set

Each $x \in \mathcal{O}$ is contained in $(x-\epsilon, x+\epsilon) \subseteq \mathcal{O}$ for some $\epsilon > 0$

Let $I_x = (\alpha, \beta)$ where $\alpha = \inf\{a : (a, x) \subseteq \mathcal{O}\}$

$\beta = \sup\{b : (x, b) \subseteq \mathcal{O}\}$



So, $\boxed{\mathcal{O} = \bigcup_{x \in \mathcal{O}} I_x}$ But why is this a countable union?

★ **Observation:** $\forall x, y \in \mathcal{O}$, $I_x = I_y$ or $I_x \cap I_y = \emptyset$

Since each I_x contains a rational #, Observation tells us there can not be more intervals than $|\mathbb{Q}|$ ★

MATH 400

Real Analysis

Video #20

Defn A point x is a **limit point** of a set A if $V_\epsilon(x) \cap (A \setminus \{x\}) \neq \emptyset \quad \forall \epsilon > 0.$

"every ϵ-neighborhood of x intersects A in something other than x."

Another way $\rightarrow$

Defn A point x is a *limit point* of a set A
if $V_\epsilon(x) \cap (A \setminus \{x\}) \neq \phi \quad \forall \epsilon > 0.$

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Theorem: x is a limit point of A iff
$x = \lim_{n\to\infty} a_n$ for some sequence $(a_n) \subseteq A - \{x\}$

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Theorem x is a limit point of A iff
$x = \lim_{n\to\infty} a_n$ for some sequence $(a_n) \subseteq A \setminus \{x\}$

Proof (Outline) $\boxed{\Rightarrow}$ Let x be a limit point of A : take $\epsilon = 1/n$
Every $1/n$-neighborhood of x intersects $A \setminus \{x\}$, so
pick $a_n \in \underbrace{V_{1/n}(x) \cap (A \setminus \{x\})}_{\neq \phi}$.
Verify $a_n \to x$.

Defn A point x is a <u>limit point</u> of a set A if $V_\epsilon(x) \cap (A \setminus \{x\}) \neq \emptyset \quad \forall \epsilon > 0.$

"every ϵ-neighborhood of x intersects A in something other than x."

<u>Another way</u> $\rightarrow$

Theorem x is a limit point of A iff

$x = \lim_{n\to\infty} a_n$ for some sequence $(a_n) \subseteq A \setminus \{x\}$

<u>Proof (Outline)</u> $\boxed{\Leftarrow}$ The definition of convergence tells us $\forall \epsilon > 0$, <u>$a_n \in V_\epsilon(x)$</u> $\forall n \geq N$, so <u>$a_N \in V_\epsilon(x)$ and $a_N \neq x$.</u>

Also, $\bar{A}$ is the smallest closed set containing A

Defn For $A \subseteq \mathbb{R}$, let L be the set of all limit points of A. The closure of A is defined to be $\bar{A} = A \cup L$.

Defn Set A is closed if $A = \bar{A}$, i.e., A contains all its limit points.

Theorem $A \subseteq \mathbb{R}$ is closed iff every Cauchy sequence in A has a limit also in A.

Proof Exercise!

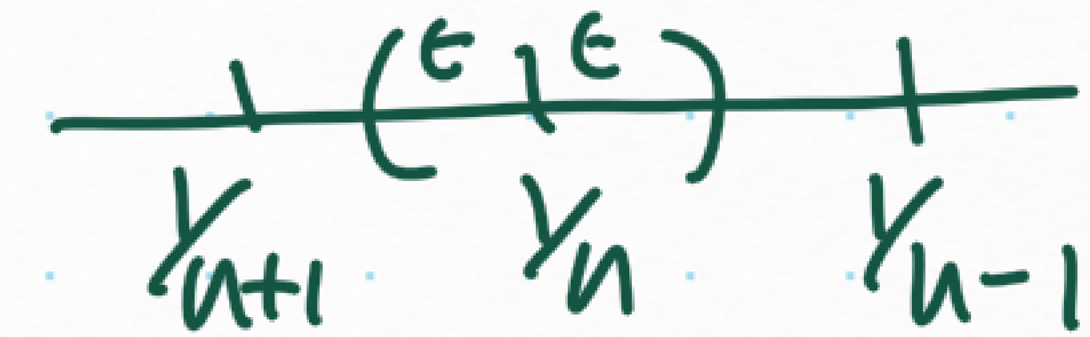
Comment "Closed" under the operation of taking limits of sequences.

not a limit point of A

Examples

① $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ Every point of A is isolated

Given $\frac{1}{n} \in A$, we need to find $\epsilon > 0$ s.t. $V_\epsilon(\frac{1}{n}) \cap A = \{1/n\}$



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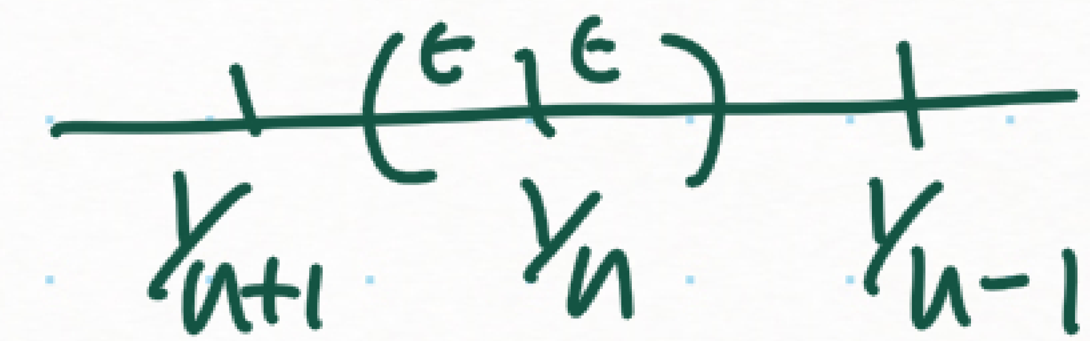

Pick $\epsilon = \frac{1}{n} - \frac{1}{n+1}$ (distance between $1/n$ & $\frac{1}{n+1}$)

Any limit points of A?

not a limit point of A

Examples

① $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ Every point of A is isolated

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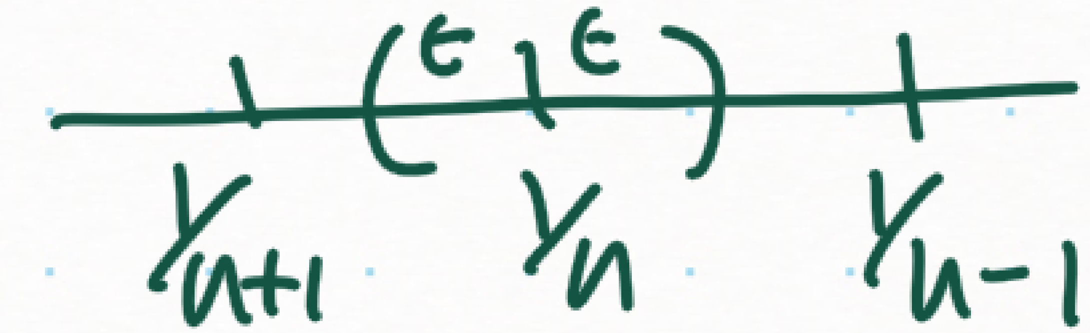

Pick $\epsilon = \frac{1}{n} - \frac{1}{n+1}$ (distance between $1/n$ & $\frac{1}{n+1}$)

Any limit points of A?

0 is the only limit point

Every $V_\epsilon(0)$ will intersect A (Why?)

$= (-\epsilon, \epsilon) \cap \{1/n : n \in \mathbb{N}\} \neq \emptyset$ $\exists n$ s.t. $1/n < \epsilon$

$\therefore \bar{A} = A \cup \{0\}$

② $[c,d]$ is a closed set

We want to prove every limit pt. of $[c,d]$ belongs to it.

If x is limit pt. of $[c,d]$ then

by Thm (x limit pt. of $A \Leftrightarrow x = \lim a_n$ for $(a_n) \subseteq A - \{x\}$),

we know $x = \lim x_n$ where $x_n \in [c,d] \setminus \{x\}$

Does $x \in [c,d]$?

since $c \le x_n \le d$, by Order limit Thm,

$c \le \lim x_n \le d$

i.e., $c \le x \le d$

i.e, $x \in [c,d]$ ✓

So, closure of $[c,d] = [c,d]$

③ Is the set $\mathbb{Q} \subseteq \mathbb{R}$ closed?

Let $x \in \mathbb{R}$ & $V_\epsilon(x) = (x-\epsilon, x+\epsilon)$ be any neighborhood of x

By Thm (Density of $\mathbb{Q}$ in $\mathbb{R}$), we know $\exists\, r \neq x$ s.t. $r \in (x-\epsilon, x+\epsilon) \cap \mathbb{Q}$

That is,

③ Is the set $\mathbb{Q} \subseteq \mathbb{R}$ closed?

Let $x \in \mathbb{R}$ & $V_\epsilon(x) = (x-\epsilon, x+\epsilon)$ be any neighborhood of x

By Thm (Density of $\mathbb{Q}$ in $\mathbb{R}$), we know $\exists\, r \neq x$ s.t. $r \in (x-\epsilon, x+\epsilon) \cap \mathbb{Q}$

That is, x is a limit point of $\mathbb{Q}$.

$$\boxed{\therefore\ \overline{\mathbb{Q}} = \mathbb{R}}$$

Theorem [Alternate form of Density of $\mathbb{Q}$ in $\mathbb{R}$]

For every $x \in \mathbb{R}$, $\exists$ seq. of rational numbers that converges to x.

(4) Is $\mathbb{R}$ closed?

(5) Is (a,b) closed? What is $\overline{(a,b)}$?

(6) Is $(a,b]$ closed? What is $\overline{(a,b]}$?

⋮

→ Are there any sets that are both open and closed?

→ Are there any sets that are neither open nor closed?

complement of F

Theorem A set F is open $\Longleftrightarrow$ F^c is closed

Since $(F^c)^c = F$, this means F is closed $\Longleftrightarrow$ F^c is open.

complement of $\mathcal{O}$

Theorem A set F is open $\Longleftrightarrow$ F^c is closed

Since $(F^c)^c = F$, this means F is closed $\Longleftrightarrow$ F^c is open.

Proof Let $F \subseteq \mathbb{R}$ be open

To show F^c is closed, we need to show it contains all its limit points.

Let x be a limit point of F^c

then $V_\epsilon(x) \cap (F^c \setminus \{x\}) \neq \phi \quad \forall \epsilon > 0$

i.e., every ϵ-neighborhood of x contains a pt. of F^c — $\circledast$

Claim $x \in F^c$

If. If $x \notin F^c$, i.e., $x \in F$ then $\exists \epsilon > 0$ s.t. $V_\epsilon(x) \subseteq F$ ($\because F$ open)

which is not possible by $\circledast$

contradiction

complement of O

Theorem A set F is open $\iff$ F^c is closed

Since $(F^c)^c = F$, this means F is closed $\iff$ F^c is open.

Proof Let F^c be closed

To show F is open, ~~for~~ each $x \in F$ we must find $\epsilon > 0$ s.t. $V_\epsilon(x) \subseteq F$

Note that $x^{\in F}$ cannot be a limit point of F^c (since F^c is closed & contains all its limit points)

By negation of defn of limit pt., we have

$\exists \epsilon > 0$ s.t. $V_\epsilon(x) \cap F^c = \emptyset$

i.e, $V_\epsilon(x) \subseteq F$, as needed.

Using this characterization & properties of open sets, we have

Theorem ① Union of a finite collection of closed sets is closed

② Intersection of an arbitrary collection of closed sets is closed.

(De Morgan's Laws: $\left(\bigcup_{\lambda} A_\lambda\right)^c = \bigcap_{\lambda} A_\lambda^c$

$\left(\bigcap_{\lambda} A_\lambda\right)^c = \bigcup_{\lambda} A_\lambda^c$)

MATH 400

Real Analysis

Video #21

Heine-Borel Theorem

Let $K \subseteq \mathbb{R}$. The following statements are equivalent and characterize compact sets in $\mathbb{R}$:

① Every sequence in K has a subsequence that converges to a limit that is also in K.

② K is closed and bounded

③ Every open cover for K has a finite subcover.

Heine-Borel Theorem

Let $K \subseteq \mathbb{R}$. The following statements are equivalent and characterize compact sets in $\mathbb{R}$:

1. Every sequence in K has a subsequence that converges to a limit that is also in K.

sequential compactness

2. K is closed and bounded

3. Every open cover for K has a finite subcover.

compactness

<u>Defn</u> $K \subseteq \mathbb{R}$ is <u>sequentially compact</u> if every sequence in K has a subseq that converges to a limit that is in K.

e.g. $[c,d]$

If $(a_n) \subseteq [c,d]$ then

Defn $K \subseteq \mathbb{R}$ is sequentially compact if every sequence in K has a subseq. that converges to a limit that is in K.

e.g. $[c,d]$

since $c \leq a_n \leq d \ \forall n$

If $(a_n) \subseteq [c,d]$ then Bolzano-Weierstrass tells us there is a convergent subseq. (a_{n_k})

Since $[c,d]$ is a closed set, the limit of $(a_{n_k}) \in [c,d]$

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since $[c,d]$ is bounded

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Since $[c,d]$ is a closed set, the limit of $(a_{n_k}) \in [c,d]$

Defn $A \subseteq \mathbb{R}$ is bounded if $\exists M > 0$ s.t. $|a| < M \ \forall a \in A$

Theorem $K \subseteq \mathbb{R}$ is sequentially compact $\iff$ K is closed and bounded

Proof Let K be sequentially compact

Suppose K is not a bounded set.

Since K is not bounded, $\exists\, x_1 \in K$ with $|x_1| > 1$,

$\exists\, x_2 \in K$ with $|x_2| > 2$, $\cdots$, $\exists\, x_n \in K$ with $|x_n| > n$

<u>Theorem</u> $K \subseteq \mathbb{R}$ is sequentially compact
$\iff$ K is closed and bounded

<u>Proof</u> Let K be sequentially compact

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Since K is sequentially compact, $\exists$ convergent subseq. (x_{n_k})

By defn. of (x_n), we have $|x_{n_k}| > n_k$, i.e. (x_{n_k}) is unbdd.
so, (x_{n_k}) can not be convergent, contradiction.

$\therefore$ K must be bounded.

K is closed, i.e. K contains all its limit points.

Let $x = \lim x_n$ where $(x_n) \subseteq K$

<u>Claim</u> $x \in K$

with limit also in K.

Since K is seq. compact, (x_n) has a cgt. subseq. (x_{n_k})

Since $x_n \to x$, x_{n_k} must also have the same limit x.

By defn. of seq. compactness, $x \in K$.

$\boxed{\Leftarrow}$ HW exercise.

Nested Compact Set Property

If $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots\cdots$

is a nested sequence of nonempty compact sets

then $\bigcap_{n=1}^{\infty} K_n$ is not empty

"compact sets" capture the essense of "Closed intervals"

Defn Let $A \subseteq \mathbb{R}$

An open cover for A is collection of open sets $\{U_\alpha ; \alpha \in S\}$ s.t. $A \subseteq \bigcup_{\alpha \in S} U_\alpha$.

If $\{U_\alpha\}_{\alpha \in S}$ has a finite subset $\{U_\alpha\}_{\alpha \in F}$ (i.e., $F \subseteq S$ and $|F| < \infty$) which is still a cover of A, i.e., $A \subseteq \bigcup_{\alpha \in F} U_\alpha$, then $\{U_\alpha\}_{\alpha \in F}$ is called a finite subcover of A.

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Defn $K \subseteq \mathbb{R}$ is a compact set if every open cover of K contains a finite subcover of K.

Examples

Consider $(0,1)$.

x

For each $x \in (0,1)$, let $\underline{U_x = \left(\frac{x}{2}, 1\right)}$

Then $\{U_x\}_{x \in (0,1)}$ is an open cover of $(0,1)$

Examples

Consider $(0,1)$.

For each $x \in (0,1)$, let $U_x = (\frac{x}{2}, 1)$

Then $\{U_x\}_{x \in (0,1)}$ is an open cover of $(0,1)$

But there is no finite subcover here.

Consider any finite collection $\{U_{x_1}, U_{x_2}, \dots, U_{x_k}\}$

Then for $x' = \min\{x_1, \dots, x_k\}$

any y s.t. $0 < y < \frac{x'}{2}$ is not in $\bigcup_{i=1}^{k} U_{x_i}$.

$\therefore$ $(0,1)$ is not compact.

Consider $[0,1]$

The same open cover as before $\{U_x = (\frac{x}{2}, 1)\}_{x \in (0,1)}$ covers every point in $[0,1]$ except 0 & 1.

So, for a fixed $\underset{=0.1}{\epsilon} > 0$, let $U_0 = (-\epsilon, \epsilon)$ & let $U_1 = (1-\epsilon, 1+\epsilon)$

Now, $\{U_0, U_1, U_x : x \in (0,1)\}$ is an open cover of $[0,1]$

Does this open cover have a finite subcover?

Consider $[0,1]$

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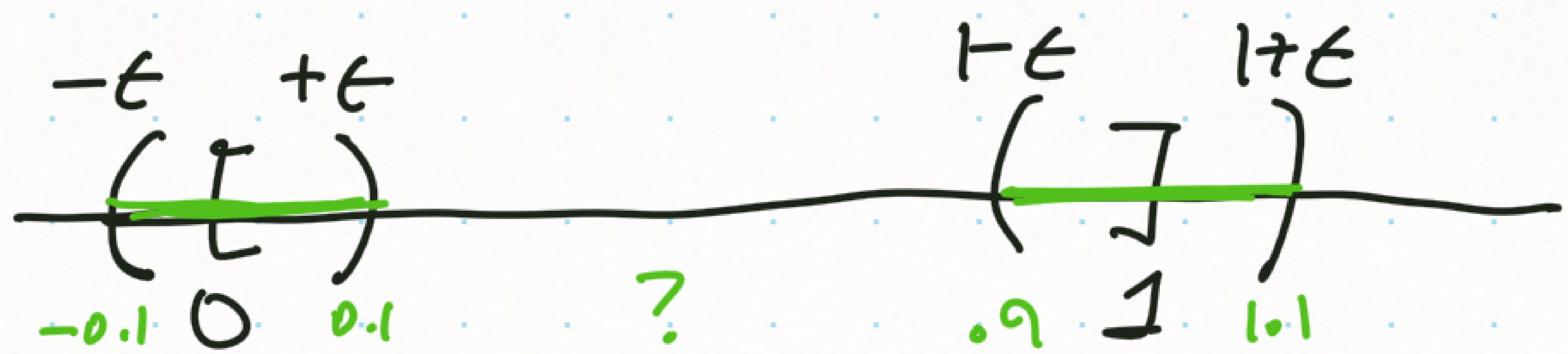

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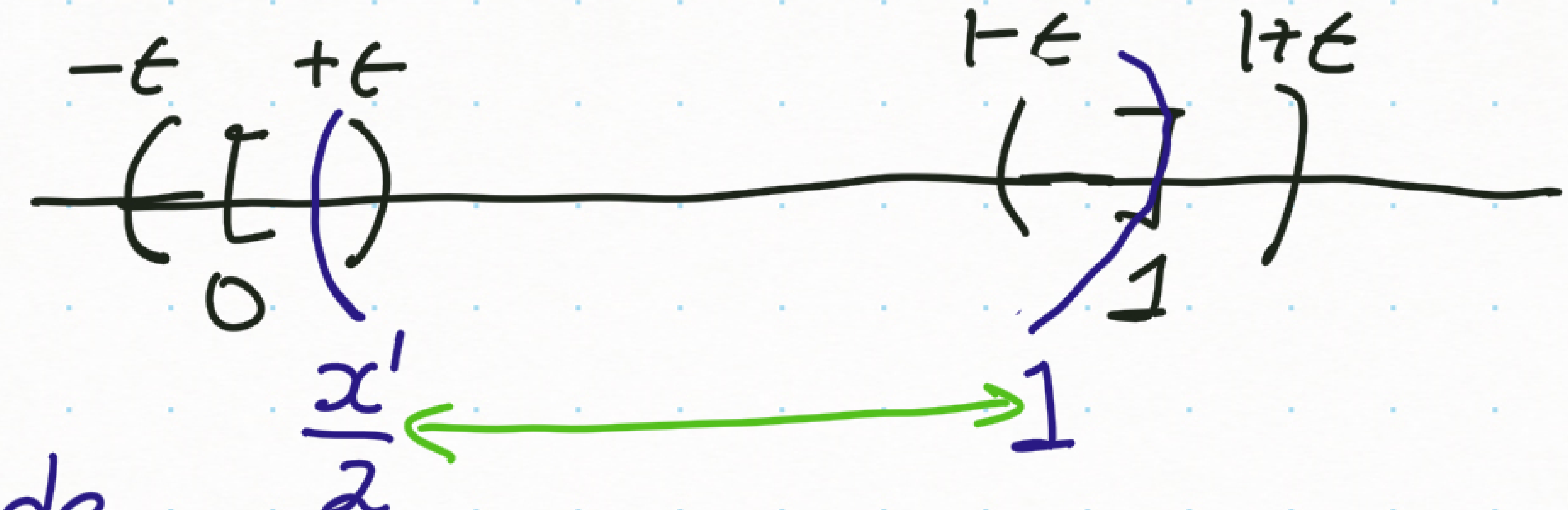

(Think of $\epsilon = 0.1$)

Also include $(\frac{x'}{2}, 1)$ where $\frac{x'}{2} < \epsilon$ (which is possible to find).

Consider $(0,4)$ and its open cover

$$\left\{\left(\frac{1}{k}, 4-\frac{1}{k}\right)\right\}_{k=1}^{\infty} = \left\{(1,3), \left(\frac{1}{2}, 4-\frac{1}{2}\right), \left(\frac{1}{3}, 4-\frac{1}{3}\right), \ldots\right\}$$

→ Check every $x \in (0,4)$ belongs to this cover

→ Check there is no finite subcover in this cover.

$\therefore (0,4)$ is not compact.

Theorem Let $K \subseteq \mathbb{R}$.

K is compact (every open cover has a finite subcover)

iff K is closed and bounded.

Proof $\boxed{\Rightarrow}$ We assume K is compact, that is every open cover of K has a finite subcover.

K is bounded

Let $I_n = (-n, n)$, then $\mathbb{R} = \bigcup_{n=1}^{\infty} I_n$

& since $K \subseteq \mathbb{R}$, $K \subseteq \bigcup_{n=1}^{\infty} I_n$, an open cover of K.

$$K \subseteq \bigcup_{n=1}^{\infty} I_n \text{ , an open cover}$$

so, there is a finite subcover $\{I_{n_1}, I_{n_2}, \ldots, I_{n_k}\}$.

Assume $n_1 < n_2 < \cdots < n_k$

then $I_{n_1} = (-n_1, n_1) \subseteq I_{n_2} = (-n_2, n_2) \subseteq \cdots\cdots \subseteq I_{n_k}$

Hence $K \subseteq \bigcup_{i=1}^{k} I_{n_i} = I_{n_k} = (-n_k, n_k)$

i.e, every element of $K \in (-n_k, n_k)$

i.e, K is bounded.

K is closed: Suppose K is not closed.

Then $\exists\, x \notin K$ and $(a_n) \subseteq K$ s.t. $\lim_{n\to\infty} a_n = x$ — ⊛

Let $U_n = (-\infty, x-\frac{1}{n}) \cup (x+\frac{1}{n}, \infty)$, open set

$\mathbb{R} - [x-\frac{1}{n}, x+\frac{1}{n}]$

and $\bigcup_{n=1}^{\infty} U_n = \mathbb{R} - \{x\}$.

Since $K \subseteq \mathbb{R} \setminus \{x\}$, $K \subseteq \bigcup_{n=1}^{\infty} U_n$, an open cover of K.

By compactness of K, $\{U_n\}_{n=1}^{\infty}$ contains a finite subcover of K: $\{U_{n_1}, U_{n_2}, \ldots, U_{n_k}\}$

Assume $n_1 < n_2 < \cdots < n_k$, & thus $K \subseteq \bigcup_{\ell=1}^{k} U_{n_\ell} = U_{n_k}$

$K \subseteq U_{n_k}$ means $K \subseteq (-\infty, x-\frac{1}{n_k}) \cup (x+\frac{1}{n_k}, \infty)$

which implies $K \cap (x-\frac{1}{n_k}, x+\frac{1}{n_k}) = \emptyset$

Why is this a problem?

there are no elements of K within distance $\frac{1}{n_k}$ of x

$K \subseteq U_{n_k}$ means $K \subseteq \left(-\infty, x - \frac{1}{n_k}\right) \cup \left(x + \frac{1}{n_k}, \infty\right)$

which implies $K \cap \left(x - \frac{1}{n_k}, x + \frac{1}{n_k}\right) = \emptyset$ — (**)

From (*) we know $(a_n) \subseteq K$ and $\lim a_n = x$

If we let $\epsilon = \frac{1}{n_k} > 0$, then (**) says

there is no a_n s.t. $|a_n - x| < \epsilon$

contradicting the definition of convergence for $\lim a_n = x$.

Math 400

Real Analysis

Video #22

What does it mean for a function to be continuous?

- Middle School → It looks like $f(x)=x^2$
- High school → You can draw it without picking up your pencil
- Pre-calculus → It does not have any holes or jumps.
- Calculus → If for each c, $\lim_{x \to c} f(x) = f(c)$

$\lim_{n\to\infty} a_n = a$

<u>Recall</u> (a_n) converges to a if

For all $\epsilon>0$, $\exists N$ s.t. $|a_n-a|<\epsilon \ \forall n>N$

How can we adapt this idea to define $\lim_{x\to c} f(x)=L$

we want $f(x)$ to be close to L when x is close to c

$|f(x)-L|<\epsilon$ $\qquad$ $|x-c|<\delta$

arbitrary $\qquad$ chosen (like N)

Recall (a_n) converges to a if
For all $\epsilon > 0$, $\exists N$ s.t. $|a_n - a| < \epsilon$ $\forall n > N$

How can we adapt this idea to define $\lim_{x \to c} f(x) = L$

we want $f(x)$ to be close to L when x is close to c

$|f(x) - L| < \epsilon$ ← arbitrary

$|x - c| < \delta$ ← chosen (like N)

Defn Let $f : A \to \mathbb{R}$ and let c be a limit pt. of A.
We say $\lim_{x \to c} f(x) = L$ if $\forall \epsilon > 0, \exists \delta > 0$ s.t.
$|f(x) - L| < \epsilon$ for every $x \in A$ which $0 < |x - c| < \delta$

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Comments

① c need not be in A (domain of f).
As long as c is a limit point of A, we can pick points in A that approach it.

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① c need not be in A (domain of f).
As long as c is a limit point of A, we can pick points in A that approach it.

② We say "for every $x \in A$ . . ." because $f(x)$ needs $x \in A$.

Let $f: A \to \mathbb{R}$ and let c be a limit pt. of A.
We say $\lim_{x\to c} f(x) = L$ if $\forall \epsilon > 0$ $\exists \delta > 0$ s.t.
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Comments

1. c need not be in A (domain of f).
   As long as c is a limit point of A, we can pick points in A that approach it.

2. We say "for every $x \in A \ldots$" because $f(x)$ needs $x \in A$.

3. $\boxed{|x-c| < \delta}$ means $x \in (c-\delta, c+\delta) = V_\delta(c)$, δ-neighborhood of c.
   $0 < |x-c|$ forces $x \neq c$, that is we don't care what happens at $x = c$

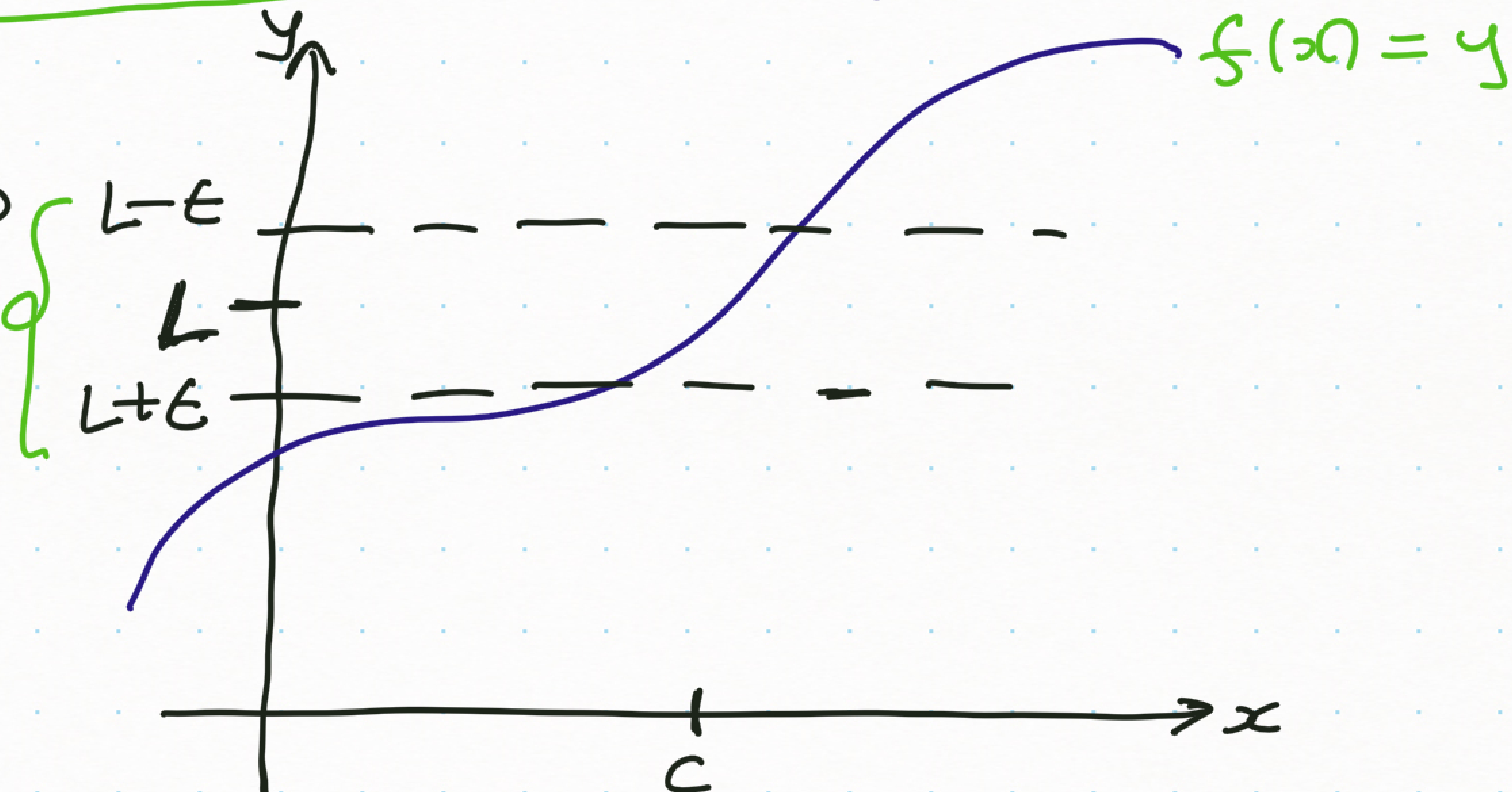
Let $f : A \to \mathbb{R}$ and let c be a limit pt. of A.

We say $\lim\limits_{x \to c} f(x) = L$ if $\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $\underline{|f(x) - L| < \epsilon}$ for every $x \in A$ which $0 < |x - c| < \delta$

4

Given $\epsilon > 0$

ϵ-neighborhood of L



Find $\delta > 0$

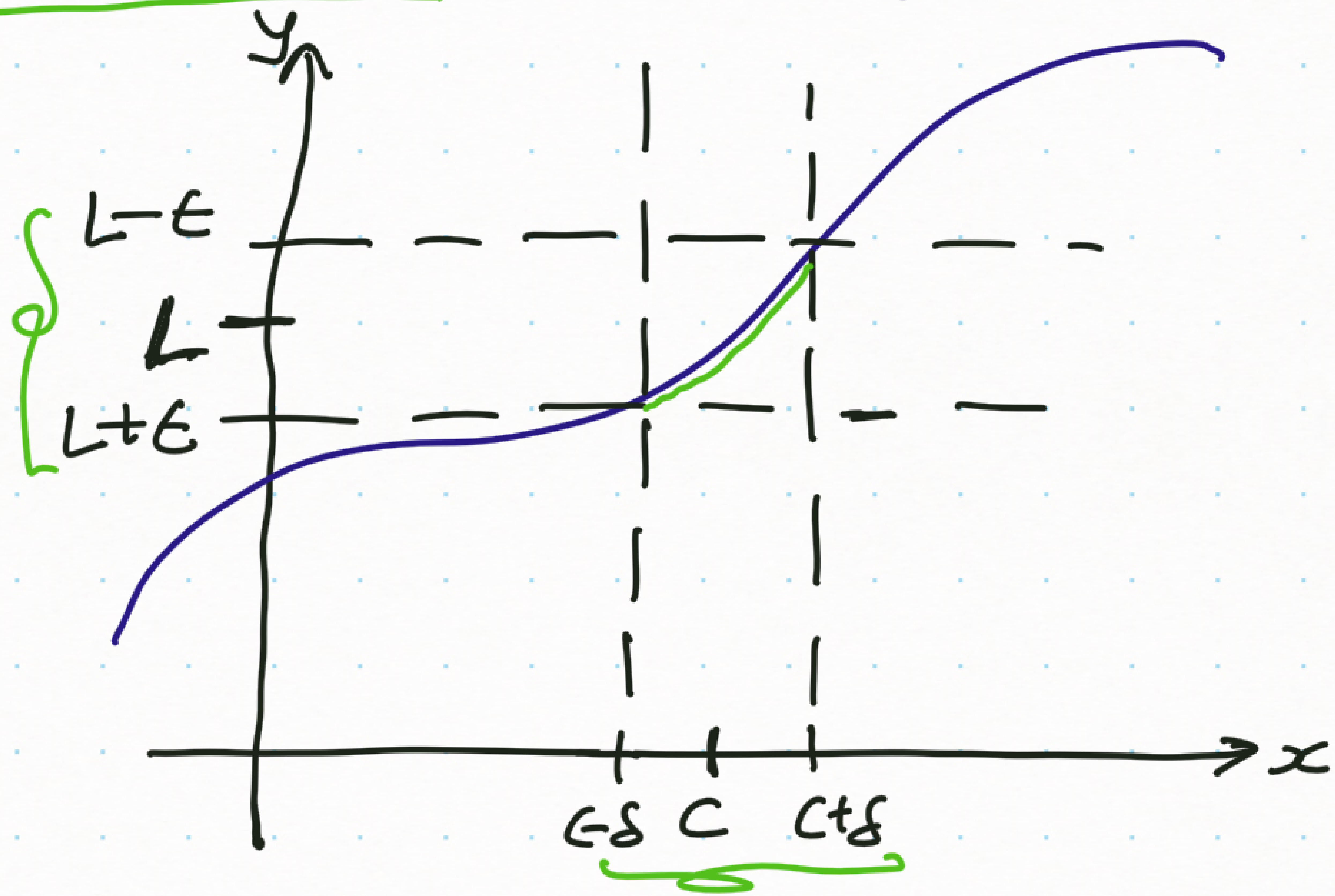
Let $f: A \to \mathbb{R}$ and let c be a limit pt. of A.

We say $\lim_{x \to c} f(x) = L$ if $\forall \epsilon > 0$ $\exists \delta > 0$ s.t.

$|f(x) - L| < \epsilon$ for every $x \in A$ which $0 < |x - c| < \delta$

(4)

Given $\epsilon > 0$



Given these horizontal lines

Find vertical lines

s.t. $f(x)$ is trapped within the box

$\exists\, \delta > 0$: $f(x) \in \epsilon$-neighborhood of L

for $x \in \delta$-neighborhood of c

⑤ To show $\lim_{x \to c} f(x)$ is not L

apply negation of definition.

$\exists \epsilon > 0$ s.t. for all $\delta > 0$, $\exists x \in A$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$.

There is an $\epsilon > 0$ s.t. we can not "trap" $f(x)$ in the "box" no matter which $\delta > 0$ we choose.

defined by ϵ-neighborhood of L and δ-neighborhood of c

Examples

① $f(x) = 5x+2$. $\lim_{x\to 3} f(x) = 17$

scratch work convert $|f(x)-L| < \epsilon$ into $|x-c| < \delta$

i.e., $|5x+2-17| < \epsilon$ i.e, $|5x-15| < \epsilon$, i.e., $5|x-3| < \epsilon$

i.e., $|x-3| < \epsilon/5$ so choose $\delta = \epsilon/5$

Soln Let $\epsilon > 0$, set $\delta = \dfrac{\epsilon}{5} > 0$ then for any x: $0 < |x-3| < \delta$

$$\begin{aligned} |f(x)-L| = |5x+2-17| = |5x-15| &= 5|x-3| \\ &< 5\delta \\ &= 5\,\frac{\epsilon}{5} \quad \text{(by choice of } \delta\text{)} \\ &= \epsilon \end{aligned}$$

② $f: \mathbb{R}-\{2\} \to \mathbb{R}$ with $f(x) = \dfrac{3x^2-12}{x-2}$. $\displaystyle\lim_{x\to 2} f(x) = 12$.

Scratch work Convert $|f(x)-L|<\epsilon$ to $|x-c|<\delta$

$|f(x)-L|<\epsilon$ i.e., $\left|\dfrac{3x^2-12}{x-2} - 12\right| < \epsilon$, i.e.,

$\left|\dfrac{3(x-2)(x+2)}{x-2} - 12\right| < \epsilon$, i.e., $|3(x+2)-12| < \epsilon$, i.e.,

$3|(x+2)-4| < \epsilon$, i.e., $3|x-2| < \epsilon$, i.e., $|x-2| < \epsilon/3$

Soln. Let $\epsilon > 0$, set $\delta = \dfrac{\epsilon}{3} > 0$,

Then for any x : $0 < |x-2| < \delta$

$|f(x)-L| = \left|\dfrac{3x^2-12}{x-2} - 12\right| = \cdots\cdots = 3|x-2|$

$< 3\delta$

$= 3\dfrac{\epsilon}{3} = \epsilon$.

③ $g(x) = x^2$. $\lim_{x \to 2} g(x) = 4$

scratch work $|x^2 - 4| < \epsilon \rightsquigarrow |x-2| < \delta$

$|x^2-4| = |(x+2)(x-2)| = |x+2|\,|x-2| < \epsilon$ compare

i.e, $|x-2| < \dfrac{\epsilon}{|x+2|}$ ~~Let $\delta = \frac{\epsilon}{|x+2|}$~~ not allowed need a number

We need to choose δ s.t.

when we do $0 < |x-2| < \delta \Rightarrow |x^2-4| = |x+2|\,|x-2|$

$< K\delta$

& we can pick $\delta = \dfrac{\epsilon}{K}$

So, we need K s.t. $|x+2| < K$ when $|x-2| < \delta$

i.e, $x < 2+\delta$

③ $g(x) = x^2$. $\lim_{x\to 2} g(x) = 4$

scratch work $\boxed{|x^2-4|<\epsilon}$ $\Leftarrow$ $\boxed{|x-2|<\delta}$

$|x^2-4| = |(x+2)(x-2)| = |x+2|\,|x-2| < \delta$ compare

i.e, $|x-2| < \dfrac{\delta}{|x+2|}$ ← not allowed need a number

we need to choose δ s.t.

when we do $0<|x-2|<\delta \Rightarrow |x^2-4| = |x+2|\,|x-2|$

$< K\,\delta$

& we can pick $\boxed{\delta = \dfrac{\epsilon}{K}}$

So, we need K s.t. $|x+2| < K$ when $x < 2+\delta$

If we ensure $\delta \leq 1$ then $|x+2| \leq |(2+1)+2| = 5$

$|x-2|<\delta$ means $x < 2+\delta \leq 2+1$

$\therefore$ choose $\boxed{\delta = \min\{1, \epsilon/5\}}$

Theorem (Sequential Criterion for Functional Limits)

Given $f: A \to \mathbb{R}$ and limit point c of A, the following are equivalent:

(i) $\boxed{\lim_{x \to c} f(x) = L.}$

(ii) For all $(x_n) \subseteq A$ with $x_n \neq c$, & $x_n \to c$, $\boxed{f(x_n) \to L.}$

sequence

defn of convergence of sequence

Use any Theorems or properties of convergent seq

Theorem (Sequential Criterion for Functional Limits)

Given $f: A \to \mathbb{R}$ and limit point c of A, the following are equivalent:

(i) $\lim_{x \to c} f(x) = L$.

(ii) $\boxed{\text{For all } (x_n) \subseteq A \text{ with } x_n \neq c, \ \& \ x_n \to c, \ f(x_n) \to L.}$

Cor Let $f, g : A \to \mathbb{R}$ with $\lim_{x \to c} f(x) = L$ & $\lim_{x \to c} g(x) = M$

(i) $\lim_{x \to c} k f(x) = kL$ for all $k \in \mathbb{R}$

(ii) $\lim_{x \to c} (f(x) + g(x)) = L + M$

(iii) $\lim_{x \to c} (f(x) g(x)) = LM$

(iv) $\lim_{x \to c} \frac{f(x)}{g(x)} = L/M$ provided $M \neq 0$