MATH 400

Real Analysis

Video #23

Theorem (Sequential Criterion for Functional Limits)

Given $f: A \to \mathbb{R}$ and limit point c of A, the following are equivalent:

(i) $\lim_{x \to c} f(x) = L$.

(ii) For all $(x_n) \subseteq A$ with $x_n \neq c$, & $x_n \to c$, $f(x_n) \to L$.

Cor Let $f, g : A \to \mathbb{R}$ with $\lim_{x\to c} f(x) = L$ & $\lim_{x \to c} g(x) = M$

(i) $\lim_{x \to c} k f(x) = kL$ for all $k \in \mathbb{R}$

(ii) $\lim_{x \to c} (f(x) + g(x)) = L + M$

(iii) $\lim_{x \to c} (f(x)\, g(x)) = LM$

(iv) $\lim_{x \to c} f(x)/g(x) = L/M$ provided $M \neq 0$

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Cor (Divergence Criterion)

Given $f: A \to \mathbb{R}$ and limit point c of A.

If $\exists$ $(x_n), (y_n) \subseteq A$ with $x_n \neq c$ and $y_n \neq c$, and $\lim x_n = c = \lim y_n$ but $\lim f(x_n) \neq \lim f(y_n)$

Then $\lim_{x \to c} f(x)$ does not exist.

Theorem (Sequential Criterion for Functional Limits)

Given $f: A \to \mathbb{R}$ and limit point c of A, the following are equivalent:

(i) $\lim_{x \to c} f(x) = L$.

(ii) For all $(x_n) \subseteq A$ with $x_n \neq c$, & $x_n \to c$, $f(x_n) \to L$.

Proof

$\boxed{\Rightarrow}$ Let $\epsilon > 0$.

$\lim_{x \to c} f(x) = L$ means $\exists \delta > 0$ s.t. $|f(x) - L| < \epsilon$ for $0 < |x - c| < \delta$ (*)

Let $(x_n) \subseteq A$ be an arbitrary seq convergent to c

Since $x_n \to c$, $\exists N$ s.t. $|x_n - c| < \delta$ $\forall n > N$ (using $\delta > 0$ as ϵ in defn. of $x_n \to c$)

(& since $x_n \neq c$, $0 < |x_n - c|$ is automatic)

$\therefore |f(x_n) - L| < \epsilon$ from (*)

$\boxed{\Leftarrow}$ Assume $f(x_n) \to L$ for every $(x_n) \subseteq A \sim \{c\}$ with $x_n \to c$

Assume $\lim_{x \to c} f(x) \neq L$, i.e.,

$\exists\, \epsilon > 0$ s.t. $\underline{\forall\, \delta > 0}$ $\exists\, x \in A$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$

Taking this $\epsilon > 0$ and setting $\delta_n = \frac{1}{n}$, we get the existence of $x_n \in A$ with $0 < \underline{|x_n - c| < \frac{1}{n}}$ and $|f(x_n) - L| \geq \epsilon$

what does this
tell us about (x_n)?

$\boxed{\Leftarrow}$ Assume $f(x_n) \to L$ for every $(x_n) \subseteq A \setminus \{c\}$ with $x_n \to c$

Assume $\lim_{x \to c} f(x) \neq L$, i.e.,

$\exists\, \epsilon > 0$ s.t. $\forall\, \delta > 0$ $\exists\, x \in A$ with $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon$

Taking this $\epsilon > 0$ and setting $\delta_n = \frac{1}{n}$, we get the existence of $x_n \in A$ with $\underline{0 < |x_n - c| < \frac{1}{n}}$ and $\underline{|f(x_n) - L| \geq \epsilon}$

Since $|x_n - c| < \frac{1}{n}$ $\forall n$, we have found a sequence $\underline{x_n \in A \setminus \{c\}}$ with $x_n \to c$.

But $|f(x_n) - L| \geq \epsilon$ $\forall n$, so $f(x_n) \not\to L$.

Contradiction.

f is continuous at a point c if $\lim_{x \to c} f(x) = f(c)$

more precisely $\to$

<u>Defn</u> $f : A \to \mathbb{R}$ is <u>continuous</u> at $c \in A$

if $\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $|f(x) - f(c)| < \epsilon \ \forall x \in A$ with $|x - c| < \delta$

If f is continuous at every point in it's domain, then f is called continuous.

Note the differences with "limit" definition

- $c \in A$
- $x = c$ allowed

Theorem [Summary of Definitions of Continuity]

Let $f : A \to \mathbb{R}$ and $c \in A$. Then the following are equivalent:

① f is continuous at c

② $\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $|f(x) - f(c)| < \epsilon \ \forall x \in A$ with $|x - c| < \delta$

③ $\forall V_\epsilon(f(c)) \ \exists V_\delta(c)$ s.t. $f(x) \in V_\epsilon(f(c)) \ \forall x \in V_\delta(c) \cap A$

④ $\lim_{x \to c} f(x) = f(c)$ if c is a limit pt. of A

⑤ $\forall (a_n) \subseteq A$ with $\lim a_n = c$, we have $\lim f(a_n) = f(c)$

Sequential characterization of continuity gives us:

Cor [Discontinuity Criterion]

① If $\exists\, (a_n) \subseteq A$ with $a_n \to c$ but $f(a_n) \not\to f(c)$ then f is discontinuous at c.

② If $\exists\, (a_n), (b_n) \subseteq A$ with $a_n \to c$ and $b_n \to c$ but $\lim f(a_n) \neq \lim f(b_n)$, then f is discontinuous at c.

Sequential characterization of continuity and algebra of functional limits gives us:

Theorem [Algebra of Continuous functions]

Let $f: A \to \mathbb{R}$ and $g: A \to \mathbb{R}$ be continuous at $c \in A$, then

① $k f(x)$ is continuous at c, for any $k \in \mathbb{R}$

② $f(x) + g(x)$ is continuous at c

③ $f(x) \cdot g(x)$ is continuous at c

④ $\frac{f(x)}{g(x)}$ is continuous at c, provided $g(x) \neq 0 \; \forall x \in A$.

Another application of sequential characterization of continuity

Theorem [f, g continuous $\Rightarrow$ $f \circ g$ continuous]

Let $g : A \to B$ and $f : B \to \mathbb{R}$.

If g is continuous at $c \in A$ and f is continuous at $g(c) \in B$

then $f \circ g : A \to \mathbb{R}$ is continuous at c.

Algebra and composition of continuous functions allow us to build complicated continuous functions from simpler ones.

- $f(x) = x$ is continuous [Verify the definition]

Algebra and composition of continuous functions allow us to build complicated continuous functions from simpler ones.

- $f(x) = x$ is continuous [Verify the definition]
- $f(x) = x^n$ is continuous for each $n \in \mathbb{N}$
- $f(x) = a_n x^n$ ———— " ———— $n \in \mathbb{N}$ & $a_n \in \mathbb{R}$
- $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k$ is continuous
- Rational function: $\dfrac{a_0 + a_1 x + \cdots + a_k x^k}{b_0 + b_1 x + \cdots + b_m x^m}$ is continuous wherever it is defined.

Algebra and composition of continuous functions allow us to build complicated continuous functions from simpler ones.

- $f(x) = \sqrt{x}$ defined on $A = \{x \in \mathbb{R} : x \geq 0\}$ is continuous

  - See Ex. 2.3.1 from HW#3 for a sequential proof
  - Give a direct ϵ-δ proof see pg. 125 of the textbook

- $f(x) = \sqrt{a_0 + a_1 x + \dots + a_k x^k}$ is continuous wherever its defined

  since $f(x) = g(h(x))$ where $g(x) = \sqrt{x}$
  $h(x) = a_0 + a_1 x + \dots + a_k x^k$
  are both continuous.

- $\lim_{x \to 0} \sin(1/x)$ does not exist

Let

$x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{2n\pi + \pi/2}$

then $x_n \to 0$ and $y_n \to 0$

But, $\sin(1/x_n) = 0$ & $\sin(1/y_n) = 1$

so, $\lim (\sin(1/x_n)) \neq \lim (\sin(1/y_n))$

- $\lim_{x \to 0} \sin(1/x)$ does not exist

Let $x_n = \frac{1}{2n\pi}$, $y_n = \frac{1}{2n\pi + \pi/2}$

then $x_n \to 0$ and $y_n \to 0$

But, $\sin(1/x_n) = 0$ & $\sin(1/y_n) = 1$

so, $\lim(\sin(1/x_n)) \neq \lim(\sin(1/y_n))$

But $g(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous at $c = 0$.

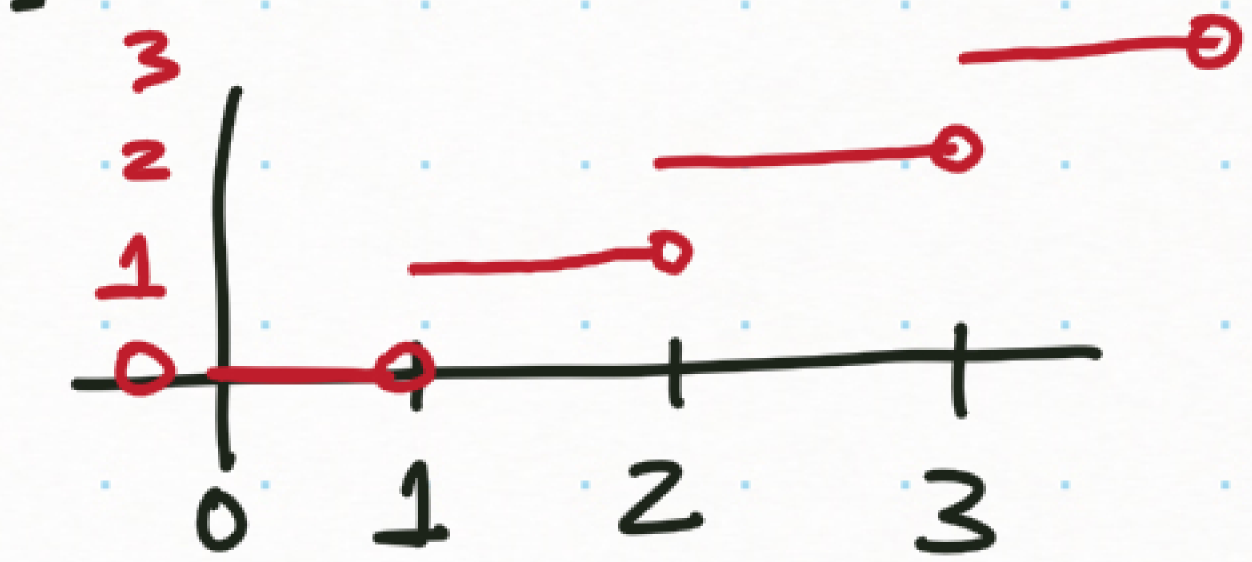
Note

$|g(x) - g(0)| = |x\sin(1/x) - 0| = |x\sin(1/x)| \leq |x|\,|\sin(1/x)| \leq |x|$

Given $\epsilon > 0$, set $\delta = \epsilon$ so $|x - 0| = |x| < \delta$

$\Rightarrow |g(x) - g(0)| \leq |x| < \epsilon$

- $h(x) = [[x]]$, largest integer at most x



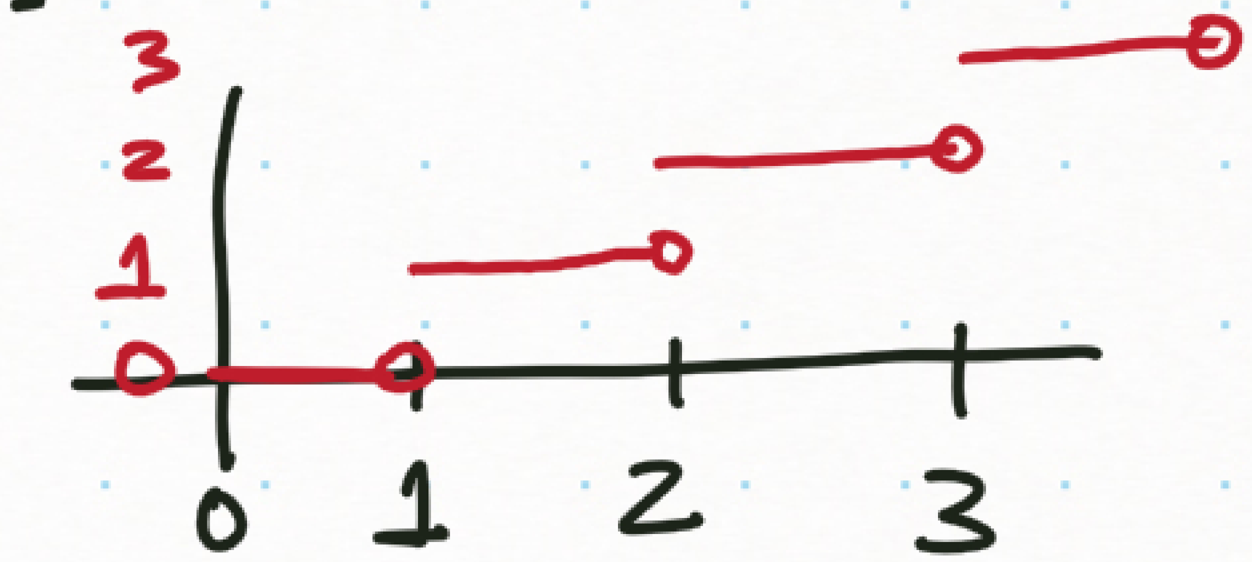
h is discontinuous at each $m \in \mathbb{Z}$

Define $x_n = m - \frac{1}{n}$. Then $x_n \to m$ as $n \to \infty$

But $h(x_n) = m-1 \to m-1$

$\neq m = h(m)$

- $h(x) = [[x]]$, largest integer at most x



h is discontinuous at each $m \in \mathbb{Z}$

Define $x_n = m - \frac{1}{n}$. Then $x_n \to m$

But $h(x_n) = m-1 \to m-1$
$\neq m = h(m)$

h is continuous at each $c \notin \mathbb{Z}$

Let $c \notin \mathbb{Z}$ Then $c \in (m, m+1)$ for some $m \in \mathbb{Z}$

Given $\epsilon > 0$, we need $\delta > 0$ s.t. $x \in V_\delta(c) \Rightarrow h(x) \in V_\epsilon(h(c))$

Let $\underline{\delta = \min\{c-m, (m+1)-c\}}$ then $h(x) = h(c)\ \forall x \in (c-\delta, c+\delta)$

& hence $h(x) \in V_\epsilon(h(c))$.

Does there exist a function that is discontinuous everywhere (continuous nowhere)?

<u>Dirichlet's Function</u>

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- Recall <u>$\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{I}$ is dense in $\mathbb{R}$</u>

- Show f is discontinuous at each $c \in \mathbb{R}$

<u>Idea:</u> Find $(a_n) \subseteq \mathbb{Q}$ s.t. $a_n \to c$ ✓

Find $(b_n) \subseteq \mathbb{I}$ s.t. $b_n \to c$ ✓

Then $f(a_n) = 1 \ \forall n$ & $f(b_n) = 0 \ \forall n$

so, $\lim f(a_n) \neq \lim f(b_n)$

Does there exist a function that is continuous at exactly one point and discontinuous everywhere else?

Modified Dirichlet's Function

$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- Show g is discontinuous at each $c \neq 0$ (construct sequences)
- Show g is continuous at $c = 0$

Can a function be continuous at every irrational and discontinuous at every rational number?

Thomae's Function

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \text{ in lowest terms} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

- Show t is discontinuous at each $c \in \mathbb{Q}$.

[Since $t(x) > 0$ for $x \in \mathbb{Q}$ while $t(x) = 0$ for $x \notin \mathbb{Q}$]

- Show t is continuous at each $c \notin \mathbb{Q}$.

But this idea doesn't work here!!

Math 400

Real Analysis

Video #24

Given $f : A \to \mathbb{R}$

For any $B \subseteq A$, $\underbrace{f(B) = \{f(x) : x \in B\}}_{\text{range of } f \text{ over } B}$

Which properties of B are preserved in $f(B)$ when f is a continuous function?

→ Open?

→ Closed?

⋮

If B is open & f is continuous then f(B) is open?

Consider $f(x)=x^2$, $B=(-1,1)$

$f(B)=[0,1)$

If B is open & f is continuous then $f(B)$ is open?

Consider $f(x) = x^2$, $B = (-1, 1)$

$f(B) = [0, 1)$ not open

If B is closed & f is continuous then $f(B)$ is closed?

Consider $g(x) = \dfrac{1}{1+x^2}$, $B = [0, \infty)$

$g(B) =$

If B is open & f is continuous then $f(B)$ is open?

Consider $f(x) = x^2$, $B = (-1, 1)$ open

$f(B) = [0, 1)$ not open

If B is closed & f is continuous then $f(B)$ is closed?

Consider $g(x) = \frac{1}{1+x^2}$, $B = [0, \infty)$ closed

$g(B) = (0, 1]$ not closed

What if B is closed and bounded?

Theorem [Preservation of Compact Sets]

Let $f : A \to \mathbb{R}$ be continuous on A.
If $K \subseteq A$ is compact then $f(K)$ is also compact

Proof We will show $f(K)$ is sequentially compact.

Let $(y_n) \subseteq f(K)$ be an arbitrary seq.

i.e., for each $n \in \mathbb{N}$, $\exists\, x_n \in K$ s.t. $f(x_n) = y_n$

$\therefore$ (x_n) is a seq in K

Since K is compact, $\exists$ subseq (x_{n_k}) with $\lim_{k\to\infty} x_{n_k} = x \in K$

Since f is continuous, we have $f(x_{n_k}) \to f(x) \in f(K)$

i.e., $\lim_{k\to\infty} y_{n_k} = f(x) \in f(K)$.

Extreme Value Theorem

If $f: K \to \mathbb{R}$ is continuous on a compact set $K \subseteq \mathbb{R}$, then f attains its maximum and minimum value in K.

In other words, $\exists\, x_0, x_1 \in K$ s.t. $f(x_0) \leq f(x) \leq f(x_1)$ $\forall x \in K$.

Extreme Value Theorem

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In other words, $\exists\, x_0, x_1 \in K$ s.t. $f(x_0) \leq f(x) \leq f(x_1)$ $\forall x \in K$.

Proof Since $f(K)$ is compact (i.e., closed & bounded), $\alpha = \sup f(K)$ exists, and $\beta = \inf f(K)$ exists

By Ex 3.3.1, $\alpha \in f(K)$ and $\beta \in f(K)$.

$\therefore$ $\exists\, x_1 \in K$ s.t. $\alpha = f(x_1)$ and $\exists\, x_0 \in K$ s.t. $\beta = f(x_0)$

so, $f(x_0) = \beta \leq f(x) \leq \alpha = f(x_1)$ $\forall x \in K$.

Some examples

① $f:[-2,1] \longrightarrow \mathbb{R}$ as $f(x)=x^2$ continuous fn. on closed int.

$f([-2,1])$ has supremum $=4$ and infimum $=0$

And $\exists -2 \in [-2,1]$ s.t. $f(-2)=4$, the sup is achieved

$\exists\ 0 \in [-2,1]$ s.t. $f(0)=0$, the inf is achieved

② $f:(-2,1] \longrightarrow \mathbb{R}$ as $f(x)=x^2$ cont. fn. on non-closed set

Again the supremum $=4$ and infimum $=0$

But there is no $c \in (-2,1]$ with $f(c)=4$.

③ $f:[0,4] \longrightarrow \mathbb{R}$ as $f(x)=\begin{cases} 2x & \text{if } x \in [0,2) \\ 2 & \text{if } x \in [2,4] \end{cases}$ not cont.

Here sup is 4 but $\nexists\ c \in [0,4]$ with $f(c)=4$.

Math 400

Real Analysis

Video #25

Bolzano's Theorem

Let f be continuous on $[a,b]$.
If $f(a)$ and $f(b)$ have different signs (positive or negative)
then $\exists\, c \in [a,b]$ s.t. $f(c)=0$ (solution of $f(x)=0$)

Proof

It is enough to prove:

$$\boxed{f(a)<0 \text{ and } f(b)>0 \Rightarrow f(c)=0 \text{ for some } c\in[a,b]}$$

[When $f(a)>0$ and $f(b)<0$

we can apply the above statement to $-f(x)$
to get $-f(a)<0$ & $-f(b)>0 \Rightarrow -f(c)=0$ for some $c\in[a,b]$

We will give a sequence $[a_i, b_i]$ of nested closed intervals s.t. $f(a_i) < 0$ & $f(b_i) > 0$ $\forall i$

Then c will be the unique pt. in $\bigcap_{i=0}^{\infty} [a_i, b_i]$.

Let $[a_0, b_0] = [a, b]$ where $f(a_0) < 0$ and $f(b_0) > 0$

bisection procedure

Divide $[a_0, b_0]$ into two intervals by its midpt. $x_0 = \frac{a_0 + b_0}{2}$

$\left.\begin{array}{l} \text{If } f(x_0) > 0 \text{ then } [a_1, b_1] = [a_0, x_0] \\ \text{If } f(x_0) < 0 \text{ then } [a_1, b_1] = [x_0, b_0] \end{array}\right\}$ so, $f(a_1) < 0$ and $f(b_1) > 0$

If $f(x_0) = 0$ then we have found $c = x_0$ s.t. $f(c) = 0$.

We will give a sequence $[a_i, b_i]$ of nested closed intervals s.t. $f(a_i) < 0$ & $f(b_i) > 0$ $\forall i$

Then c will be the unique pt. in $\bigcap_{i=0}^{\infty} [a_i, b_i]$.

Let $[a_0, b_0] = [a, b]$ where $f(a_0) < 0$ and $f(b_0) > 0$

Divide $[a_0, b_0]$ into two intervals by its midpt. $x_0 = \frac{a_0 + b_0}{2}$

If $f(x_0) > 0$ then $[a_1, b_1] = [a_0, x_0]$ ⎫ so, $f(a_1) < 0$
If $f(x_0) < 0$ then $[a_1, b_1] = [x_0, b_0]$ ⎭ and $f(b_1) > 0$

If $f(x_0) = 0$ then we have found $c = x_0$ s.t. $f(c) = 0$.

For $n \geq 0$, divide $[a_n, b_n]$ (where $f(a_n) < 0$ & $f(b_n) > 0$) into two intervals by its midpoint $x_n = \frac{a_n + b_n}{2}$.

If $f(x_n) > 0$ then $[a_{n+1}, b_{n+1}] = [a_n, x_n]$ ⎫ so, $f(a_{n+1}) < 0$
If $f(x_n) < 0$ then $[a_{n+1}, b_{n+1}] = [x_n, b_n]$ ⎭ and $f(b_{n+1}) > 0$

If $f(x_n) = 0$ then we have found $c = x_n$ s.t. $f(c) = 0$.

For $n \geq 0$, we have $[a_n, b_n]$ with $f(a_n) < 0$
and $f(b_n) > 0$

Divide $[a_n, b_n]$ into two intervals by its midpoint $x_n = \frac{a_n + b_n}{2}$

$$\left.\begin{array}{l} \text{If } f(x_n) > 0 \text{ then } [a_{n+1}, b_{n+1}] = [a_n, x_n] \\ \text{If } f(x_n) < 0 \text{ then } [a_{n+1}, b_{n+1}] = [x_n, b_n] \end{array}\right\} \begin{array}{l} \text{so } f(a_{n+1}) < 0 \\ \& \ f(b_{n+1}) > 0 \end{array}$$

If $f(x_n) = 0$ then we have found $c = x_n$ s.t. $f(c) = 0$.

If $f(x_n) \neq 0$ $\forall n$ then we have $[a,b] \supseteq [a_1, b_1] \supseteq \dots \supseteq [a_n, b_n] \supseteq \dots$ an infinite sequence of nested closed intervals.

By Nested Interval Property, $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \phi$, so $\exists\, c \in [a_n, b_n]$ $\forall n$

For $n \geq 0$, we have $[a_n, b_n]$ with $f(a_n) < 0$
and $f(b_n) > 0$

Divide $[a_n, b_n]$ into two intervals by its midpoint $x_n = \frac{a_n + b_n}{2}$

$\left.\begin{array}{l} \text{If } f(x_n) > 0 \text{ then } [a_{n+1}, b_{n+1}] = [a_n, x_n] \\ \text{If } f(x_n) < 0 \text{ then } [a_{n+1}, b_{n+1}] = [x_n, b_n] \end{array}\right\}$ so $f(a_{n+1}) < 0$ & $f(b_{n+1}) > 0$

If $f(x_n) = 0$ then we have found $c = x_n$ s.t. $f(c) = 0$.

If $f(x_n) \neq 0$ $\forall n$ then we have $[a, b] \supseteq [a_1, b_1] \supseteq \cdots \supseteq [a_n, b_n] \supseteq \cdots$
an infinite sequence of nested closed intervals.

By Nested Interval Property, $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \phi$, so $\exists$ $c \in [a_n, b_n]$ $\forall n$

we have sequences (a_n) & (b_n) with $f(a_n) < 0$ & $f(b_n) > 0$ $\forall n$

And, $b_n - a_n \to 0$. $\therefore$ $\lim a_n = c = \lim b_n$

Since f is continuous, $\lim f(a_n) = f(c) = \lim f(b_n)$.

$\left.\begin{array}{l} \text{As } f(a_n) < 0, \lim f(a_n) \leq 0 \text{ \& hence } f(c) \leq 0 \\ \text{As } f(b_n) > 0, \lim f(b_n) \geq 0 \text{ \& hence } f(c) \geq 0 \end{array}\right\} \Rightarrow f(c) = 0.$

Intermediate Value Theorem

Let $f:[a,b]\to\mathbb{R}$ be continuous.
If $L\in\mathbb{R}$ s.t. $f(a)\leq L\leq f(b)$ or $f(a)\geqslant L\geqslant f(b)$
then $\exists\, c\in[a,b]$ s.t. $f(c)=L$.

"If $f:[a,b]\to\mathbb{R}$ is continuous then f takes on all values between $f(a)$ and $f(b)$"

Its enough to prove

$$\boxed{f(a)\leq L\leqslant f(b)\implies \exists\, c\in[a,b] \text{ s.t. } f(c)=L}$$

When $f(a)\geqslant L\geqslant f(b)$, we can apply the above statement to $-f$ since $f(a)\geqslant L\geqslant f(b)\Rightarrow -f(a)\leq -L\leq -f(b)$
& hence $\exists\, c$ s.t. $-f(c)=-L$

Intermediate Value Theorem

Let $f : [a,b] \to \mathbb{R}$ be continuous.
If $L \in \mathbb{R}$ s.t. $f(a) \leq L \leq f(b)$ or $f(a) \geq L \geq f(b)$
then $\exists\, c \in [a,b]$ s.t. $f(c) = L$.

Proof Assume $f(a) \leq L \leq f(b)$.
If $f(a) = L$ or $f(b) = L$ then we are done.
Else, $f(a) < L < f(b)$.
Define $g(x) = f(x) - L$. Then $g(a) = f(a) - L < 0$
and $g(b) = f(b) - L > 0$
Note g is continuous.
So, we can apply Bolzano's thm, to get
$\exists\, c \in [a,b]$ s.t. $g(c) = 0$, i.e, $f(c) - L = 0$, i.e., $f(c) = L$.

note: Bolzano's thm is special case of IVT for $L = 0$, but....

Example

$e^x - 3x = 0$ has at least two positive solutions.

why?

Example

$e^x - 3x = 0$ has at least two positive solutions.

why?

Let $f(x) = e^x - 3x$, a continuous function on $\mathbb{R}$

$f(0) = 1$, $f(1) = e - 3 < 0$, $f(2) = e^2 - 6 > 0$

$\therefore$ Bolzano's thm $\Rightarrow$ $\exists$ solution in $[0, 1]$

and $\exists$ solution in $[1, 2]$

Example

For every $a \in [-1, 1]$, $\sin x = a$ has a solution between $-\pi/2$ and $\pi/2$

since $\sin(-\frac{\pi}{2}) = -1$ and $\sin(\pi/2) = 1$,

by IVT, for each $a \in [-1, 1]$ $\exists\, c \in [-\pi/2, \pi/2]$ s.t. $\sin c = a$.

Example

For any given $n \in \mathbb{N}$,

every positive real number has a positive n^{th} root.

That is, for every $a > 0$, $\exists\, b > 0$ s.t. $b^n = a$

n^{th} positive root of a.

Proof as HW

Apply IVT.

MATH 400

Real Analysis

Video #26

Lets recall a few ϵ-δ proofs of continuity of f at c

① $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$

In main step of the proof, we estimate

$|f(x) - f(c)| = |x - c| < \epsilon$ for $|x - c| < \delta$ when $\boxed{\delta = \epsilon}$

② $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 5x + 2$

$|f(x) - f(c)| = |5x + 2 - (5c + 2)| = |5x - 5c| = 5|x - c| < \epsilon$

for $|x - c| < \delta$ when $\boxed{\delta = \frac{\epsilon}{5}}$

In both these examples, the choice of δ does <u>not</u> depend on c, the point at which continuity is desired.

③ $f: (0,\infty) \to \mathbb{R}$ with $f(x) = x^2$.

$|f(x) - f(c)| = |x^2 - c^2| = |x-c||x+c| < \epsilon$ for $|x-c| < \delta$

if $\delta < \frac{\epsilon}{|x+c|}$

cannot depend on the unknown variable x

When $\delta < 1$, then $|x-c| < 1 \Rightarrow |x+c| < 2c+1$

so we can choose $\delta < \frac{\epsilon}{2c+1}$ in that situation.

③ $f : (0,\infty) \to \mathbb{R}$ with $f(x) = x^2$.

$|f(x)-f(c)| = |x^2 - c^2| = |x-c||x+c| < \epsilon$ for $|x-c|<\delta$

if $\delta < \dfrac{\epsilon}{|x+c|}$ cannot depend on the unknown variable x

When $\delta < 1$, then $|x-c| < 1 \Rightarrow |x+c| < 2c+1$

so we can choose $\delta < \dfrac{\epsilon}{2c+1}$ in that situation.

$\therefore$ the formal proof says.

Given $\epsilon > 0$, choose $\boxed{\delta = \min\{1, \frac{\epsilon}{2c+1}\}} > 0$

s.t. $|x-c| < \delta \Rightarrow |f(x) - f(c)| \leq \cdots\cdots < \epsilon$

- choice of δ depends on c, the point at which continuity is desired.
- $\delta = \delta(\epsilon, c)$ because as c becomes larger we need smaller & smaller δ, its choice inversely proportional to c.

④ $f: [1\ 4] \to \mathbb{R}$ with $f(x) = x^2$

$|f(x) - f(c)| = |x^2 - c^2| = |x+c|\,|x-c| \leq 8\,|x-c|$ (since $x, c \in [1,4] \Rightarrow |x+c| \leq 8$)

$< \epsilon$

whenever $|x-c| < \delta$

for $\boxed{\delta = \frac{\epsilon}{8}}$

- Same function but on a different domain leads to $\delta = \delta(\epsilon)$, that does not depend on c.

We say a function is uniformly continuous if we can choose a $\delta = \delta(\epsilon)$ that does not depend on the point c.

Definition Let $f: A \to \mathbb{R}$. f is uniformly continuous on A if $\forall \epsilon > 0$ $\exists \delta > 0$ s.t. for all $x, y \in A$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$$

Some examples of (uniformly?) continuous functions:

① $f(x) = 5x+2$ on $\mathbb{R}$ $\delta = \varepsilon/5$ works Uniformly Cont. on $\mathbb{R}$

Some examples of (uniformly?) continuous functions:

① $f(x) = 5x+2$ on $\mathbb{R}$ $\quad \delta = \epsilon/5$ works $\quad$ Uniformly Cont. on $\mathbb{R}$

② $f(x) = x^2$ on $(0,\infty)$ $\quad \delta = \delta(\epsilon, c) = \min\{1, \frac{\epsilon}{2c+1}\}$ $\quad$ Not U. Cont.?

seems unavoidable

③ $f(x) = x^2$ on $[1,4]$ $\quad \delta = \frac{\epsilon}{8}$ works $\quad$ Unif. Cont. on $[1,4]$

Some examples of (uniformly?) continuous functions:

1. $f(x) = 5x+2$ on $\mathbb{R}$ $\delta = \epsilon/5$ works   Uniformly Cont. on $\mathbb{R}$
2. $f(x) = x^2$ on $(0,\infty)$ $\delta = \delta(\epsilon, c) = \min\{1, \frac{\epsilon}{2c+1}\}$ seems unavoidable   Not U. Cont.?
3. $f(x) = x^2$ on $[1,4]$ $\delta = \frac{\epsilon}{8}$ works   Unif. Cont. on $[1,4]$
4. $f(x) = \sqrt{x}$ on $(0,\infty)$ $\delta = \epsilon^2$ works   Unif. Cont. on $\mathbb{R}^+$
5. $f(x) = \sin x$ on $\mathbb{R}$ $\delta = \epsilon$ works   Unif. Cont. on $\mathbb{R}$
6. $f(x) = \cos x$ on $\mathbb{R}$ $\delta = \epsilon$ works   Unif. Cont. on $\mathbb{R}$

Some examples of (uniformly?) continuous functions:

① $f(x) = 5x+2$ on $\mathbb{R}$ $\quad \delta = \epsilon/5$ works $\quad$ Uniformly Cont. on $\mathbb{R}$

② $f(x) = x^2$ on $(0,\infty)$ $\quad \delta = \delta(\epsilon, c) = \min\{1, \frac{\epsilon}{2c+1}\}$ seems unavoidable $\quad$ Not U. Cont.?

③ $f(x) = x^2$ on $[1,4]$ $\quad \delta = \frac{\epsilon}{8}$ works $\quad$ Unif. Cont. on $[1,4]$

④ $f(x) = \sqrt{x}$ on $(0,\infty)$ $\quad \delta = \epsilon^2$ works $\quad$ Unif. Cont. on $\mathbb{R}^+$

⑤ $f(x) = \sin x$ on $\mathbb{R}$ $\quad \delta = \epsilon$ works $\quad$ Unif. Cont. on $\mathbb{R}$

⑥ $f(x) = \cos x$ on $\mathbb{R}$ $\quad \delta = \epsilon$ works $\quad$ Unif. Cont. on $\mathbb{R}$

⑦ $f(x) = 1/x$ on $(0,\infty)$ $\quad \delta = \delta(\epsilon, c)$ seems unavoidable $\quad$ Not Unif. Cont.?

⑧ $f(x) = 1/x$ on $(3,\infty)$ $\quad \delta = \delta(\epsilon)$ can be found $\quad$ Unif. Cont.

How to show f is not Unif. Cont. on A?

Prove the negation of the definition of Unif. Cont.

Theorem [Sequential Criterion for Non-Unif. Cont.]

$f: A \to \mathbb{R}$ fails to be Unif. Cont. on A if and only if $\exists\ \epsilon_0 > 0$ and $(x_n), (y_n) \subseteq A$ such that

$$|x_n - y_n| \to 0 \quad \text{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0$$

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Proof. f not unif cont. on $A \iff \exists\, \epsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists\, x, y \in A$ s.t. $|x - y| < \delta$ but $|f(x) - f(y)| \geq \epsilon_0$.

Set $\delta_n = \frac{1}{n}$ $\forall n \in \mathbb{N}$, so $\exists\, x_n, y_n \in A$ s.t. $|x_n - y_n| < \frac{1}{n}$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$.

<u>example</u> $h(x) = \sin(\frac{1}{x})$ is continuous on $(0,1)$
But not unifomly cont. on $(0,1)$

Take $\epsilon_0 = 2$ & set $x_n = \dfrac{1}{\pi/2 + 2n\pi}$, $y_n = \dfrac{1}{3\pi/2 + 2n\pi}$

Both x_n & $y_n \to 0$, so $|x_n - y_n| \to 0$

and $|h(x_n) - h(y_n)| = |\sin(\frac{\pi}{2} + 2n\pi) - \sin(\frac{3\pi}{2} + 2n\pi)| = 2$
$\forall n \in \mathbb{N}$.

Theorem [Uniform Continuity on Compact Sets]

If $f: A \to \mathbb{R}$ is continuous on A and A is compact then f is uniformly continuous on A.

Proof

See the textbook proof that uses "Sequential Compactness" of A.

Assuming f is not unif. cont.,
using the previous theorem we get (x_n) & (y_n)
s.t. $|x_n - y_n| \to 0$ but $|f(x_n) - f(y_n)| \geq \epsilon_0$
But sequential compactness of A applied to (x_n)
& the continuity of f will lead to $|f(x_n) - f(y_n)| \to 0$.

Proof (using open cover defn. of compactness)

Let f be continous on compact set A.

Given $\epsilon > 0$, for each $c \in A$, since f is cont. on c,

$\exists\, \delta = \delta(c) > 0$ s.t. $x \in A$ and $|x - c| < \delta(c)$

to indicate δ might depend on c

$\Rightarrow |f(x) - f(c)| < \frac{\epsilon}{2}$

Note $\left\{ \left(c - \frac{\delta(c)}{2}, c + \frac{\delta(c)}{2} \right) : c \in A \right\}$ forms an open cover of A

By compactness of A, there is a finite subcover,

$A \subseteq \left\{ \left(c_1 - \frac{\delta(c_1)}{2}, c_1 + \frac{\delta(c_2)}{2} \right), \left(c_2 - \frac{\delta(c_2)}{2}, c_2 + \frac{\delta(c_2)}{2} \right), \ldots, \left(c_k - \frac{\delta(c_k)}{2}, c_k + \frac{\delta(c_k)}{2} \right) \right\}$

Set $\delta = \min \left\{ \frac{\delta(c_1)}{2}, \frac{\delta(c_2)}{2}, \ldots, \frac{\delta(c_k)}{2} \right\}$

note δ is not dependent on arbitrary c, it just a fn. of ϵ.

$$\delta = \min\left\{ \frac{\delta(c_1)}{2}, \frac{\delta(c_2)}{2}, \dots, \frac{\delta(c_k)}{2} \right\}$$

For any $x, y \in A$, since we have a finite subcover of A

$$x \in \left(c_i - \frac{\delta(c_i)}{2}, c_i + \frac{\delta(c_i)}{2}\right) \text{ for some } i \in \{1, \dots, k\}$$

i.e., $|c_i - x| < \frac{\delta(c_i)}{2}$

Also, $|x-y| < \delta \leq \frac{\delta(c_i)}{2}$.

$$\Rightarrow |c_i - y| = |c_i - x + x - y| \leq |c_i - x| + |x-y| \leq \frac{\delta(c_i)}{2} + \frac{\delta(c_i)}{2} = \delta(c_i)$$

So,

$|c_i - y| \leq \delta(c_i)$.

$$\delta = \min\left\{ \frac{\delta(c_1)}{2}, \frac{\delta(c_2)}{2}, \ldots, \frac{\delta(c_k)}{2} \right\}$$

For any $x, y \in A$, since we have a finite subcover of A

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So,

$|c_i - y| \leq \delta(c_i)$.

Since both $|x - c_i| \leq \delta(c_i)$ & $|y - c_i| \leq \delta(c_i)$,

by definition of continuity of f at c_i using $\delta(c_i)$, we have

$$|f(x) - f(c_i)| < \epsilon/2 \quad \text{and} \quad |f(y) - f(c_i)| < \epsilon/2$$

$$\therefore |f(x) - f(y)| = |f(x) - f(c_i) + f(c_i) - f(y)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

A Mathematical Theme $\rightarrow$

Local Property $\Rightarrow$ Global Property

If f is a continuous function on a compact interval

then
- f is bounded
- f has a maximum & a minimum
- f is uniformly continuous

"Local property of f on $[a,b]$" $\Rightarrow$ "Global property of f on $[a,b]$"

MATH 400

Real Analysis

Video # 27

Definition [Differentiability]

Let $f: I \to \mathbb{R}$ be a function on an interval I.

Given $c \in I$, the derivative of f at c is

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c},$$ provided the limit exists.

We say f is differentiable at c.

If f is differentiable at every $c \in I$, then f is differentiable on I.

Examples

① $f(x) = x^n$

Let $c \in \mathbb{R}$

$$f'(c) = \lim_{x \to c} \frac{x^n - c^n}{x - c} = \lim_{x \to c} \left(x^{n-1} + c x^{n-2} + c^2 x^{n-3} + \dots + c^{n-1}\right)$$

$$= c^{n-1} + c^{n-1} + \dots + c^{n-1} = n c^{n-1}$$

Examples

① $f(x) = x^n$

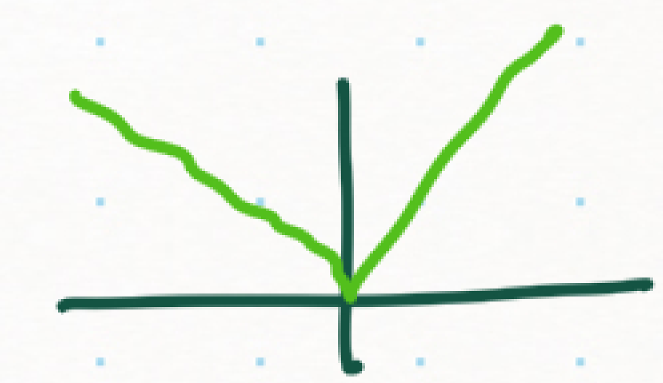
Let $c \in \mathbb{R}$

$x^n - c^n = (x-c)(x^{n-1} + cx^{n-2} + \cdots + c^{n-1})$

$$f'(c) = \lim_{x \to c} \frac{x^n - c^n}{x - c} = \lim_{x \to c} (x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1})$$

$$= c^{n-1} + c^{n-1} + \cdots + c^{n-1} = n c^{n-1}$$

② $f(x) = |x|$, let $c = 0$



Note f is continuous at 0 since $\lim_{x \to 0} f(x) = f(0)$

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{|x|}{x}$$

This limit does not exist

Examples

① $f(x) = x^n$

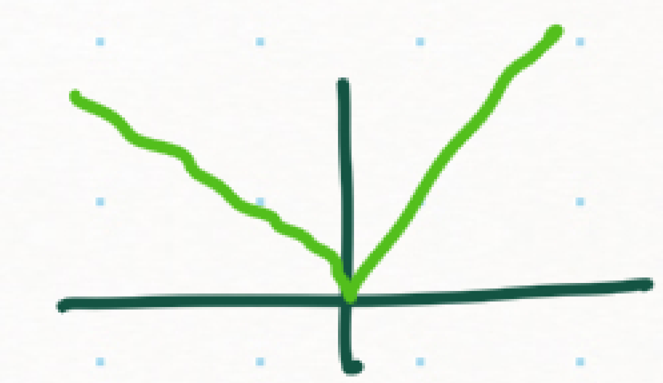
Let $c \in \mathbb{R}$

$x^n - c^n = (x-c)(x^{n-1} + cx^{n-2} + \cdots + c^{n-1})$

$$f'(c) = \lim_{x \to c} \frac{x^n - c^n}{x - c} = \lim_{x \to c} (x^{n-1} + cx^{n-2} + c^2x^{n-3} + \cdots + c^{n-1})$$

$$= c^{n-1} + c^{n-1} + \cdots + c^{n-1} = n c^{n-1}$$

② $f(x) = |x|$, let $c = 0$



Note f is continuous at 0 since $\lim_{x \to 0} f(x) = f(0)$

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \frac{|x|}{x}$$

This limit does not exist

$(a_n) = \left(\frac{1}{n}\right) \to 0$

$(b_n) = \left(-\frac{1}{n}\right) \to 0$

But $\frac{|a_n|}{a_n} = 1 \to 1$
$\neq$
$\frac{|b_n|}{b_n} = -1 \to -1$

$\Longrightarrow$

We just saw Continuity need not imply differentiability.

But

<u>Theorem</u> If $f : I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.

We just saw Continuity need not imply differentiability.
But

Theorem If $f : I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c.

Proof We assume $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ exists.

We want to show $\lim_{x \to c} f(x) = f(c)$.

By Alegbra of functional limits,

$$\lim_{x \to c} (f(x) - f(c)) = \lim_{x \to c} \left(\frac{f(x) - f(c)}{x - c} \right)(x - c) = \lim \left(\frac{f(x) - f(c)}{x - c} \right) \lim (x - c)$$

$$= f'(c)\,(0) = 0$$

$\therefore \lim_{x \to c} f(x) = f(c)$

Algebra of Differentiable Functions

Let f, g be functions on an interval I. and assume f and g are differentiable at $c \in I$. Then,

① [Linearity-1] $(f+g)'(c) = f'(c) + g'(c)$

② [Linearity-2] $(kf)'(c) = kf'(c)$ for all $k \in \mathbb{R}$.

③ [Product Rule] $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

④ [Quotient Rule] $\left(f/g\right)'(c) = \dfrac{g(c)f'(c) - f(c)g'(c)}{(g(c))^2}$, $\forall\, g(c) \neq 0$

Proof of $(f+g)'(c) = f'(c) + g'(c)$

$$\lim_{x \to c} \frac{(f+g)(x) - (f+g)(c)}{x-c} = \lim_{x \to c} \frac{[f(x)+g(x)] - [f(c)+g(c)]}{x-c}$$

$$= \lim_{x \to c} \frac{[f(x)-f(c)] + [g(x)-g(c)]}{x-c}$$

allowed since each limit exists

$$= \lim_{x \to c} \frac{f(x)-f(c)}{x-c} + \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$$

$$= f'(c) + g'(c)$$

Proof $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$

$$\frac{(fg)(x) - (fg)(c)}{x-c} = \frac{f(x)g(x) - f(c)g(c)}{x-c}$$

$$= \frac{f(x)g(x) - f(x)g(c) + f(x)g(c) - f(c)g(c)}{x-c}$$

$$= f(x)\left[\frac{g(x)-g(c)}{x-c}\right] + g(c)\left[\frac{f(x)-f(c)}{x-c}\right]$$

When we take the limit $x \to c$

$f(x) \downarrow f(c)$ since f is continuous at c (f is differentiable at c $\Rightarrow$)

$\left[\frac{g(x)-g(c)}{x-c}\right] \downarrow g'(c)$ assumed

$g(c) = g(c)$ fixed

$\left[\frac{f(x)-f(c)}{x-c}\right] \downarrow f'(c)$ assumed

Theorem [Chain Rule] Let $f: I_1 \to \mathbb{R}$ and $g: I_2 \to \mathbb{R}$ such that $f(I_1) \subseteq I_2$ so that $g \circ f$ is defined. If f is differentiable at $c \in I_1$ and g is differentiable at $f(c) \in I_2$ then $(g \circ f)'(c) = g'(f(c))\, f'(c)$.

Proof

Theorem [Chain Rule] Let $f: I_1 \to \mathbb{R}$ and $g: I_2 \to \mathbb{R}$ such that $f(I_1) \subseteq I_2$ so that $g \circ f$ is defined. If f is differentiable at $c \in I_1$ and g is differentiable at $f(c) \in I_2$ then $(g \circ f)'(c) = g'(f(c))\, f'(c)$.

Proof Consider the function
$$h(y) = \begin{cases} \dfrac{g(y) - g(f(c))}{y - f(c)} & \text{if } y \neq f(c) \\ g'(f(c)) & \text{if } y = f(c) \end{cases}$$

take lim $y \to f(c)$

Note h is continuous at $f(c)$, i.e., $\lim_{y \to f(c)} h(y) = g'(f(c))$ since g is differentiable at $f(c)$.

Claim $\dfrac{g(f(x)) - g(f(c))}{x-c} = h(f(x))\,\dfrac{f(x)-f(c)}{x-c}$

If $f(x) \neq f(c)$ then

$$\frac{g(f(x)) - g(f(c))}{x-c} = \overbrace{\frac{g(f(x)) - g(f(c))}{f(x)-f(c)}}^{h(f(x))}\,\frac{f(x)-f(c)}{x-c}$$

If $f(x) = f(c)$ then

$$\text{LHS} = \frac{g(f(x)) - g(f(c))}{x-c} = \frac{0}{x-c} = 0$$

$$\text{RHS} = g'(f(c))\,\frac{f(x)-f(c)}{x-c} = g'(f(c))\,\frac{0}{x-c} = 0$$

Claim $\dfrac{g(f(x)) - g(f(c))}{x-c} = h(f(x))\,\dfrac{f(x)-f(c)}{x-c}$

To finish the proof.

$$(g\circ f)'(c) = \lim_{x\to c} \frac{g(f(x)) - g(f(c))}{x-c}$$

$$= \lim_{x\to c} h(f(x))\,\frac{f(x)-f(c)}{x-c}$$

$$= \left(\lim_{x\to c} h(f(x))\right)\left(\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right)$$

$$= g'(f(c))\, f'(c).$$

Topologist's sine curve

$$g_n(x) = \begin{cases} x^n \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

① $$g_0(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

we have discussed $g_0(x)$ is not continuous at 0.

& hence not differentiable at 0

② $g_1(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Since $\sin(1/x) \in [-1, 1]$, we have $\underset{\downarrow\, 0}{-|x|} \leq x\sin(1/x) \leq \underset{\downarrow\, 0}{|x|}$ as $x \to 0$

By Squeeze Thm.,

$$\lim_{x\to 0} x\sin(1/x) = 0$$

$\therefore \lim_{x\to 0} g_1(x) = g_1(0)$

& g_1 is continuous at 0

But g_1 is not differentiable at 0

$$\lim_{x\to 0} \frac{g_1(x) - g_1(0)}{x - 0} = \lim_{x\to 0} \frac{x\sin(1/x) - 0}{x - 0} = \lim_{x\to 0} \sin(1/x)$$ which does not exist

③ $g_2(x) = \begin{cases} x^2 \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Same argument as $g_1(x)$ shows $g_2(x)$ is continuous at 0

g_2 is also differentiable at 0:

$$g_2'(0) = \lim_{x\to 0} \frac{g_2(x) - g_2(0)}{x - 0} = \lim_{x\to 0} \frac{x^2 \sin(1/x)}{x} = \lim_{x\to 0} x \sin(1/x) = 0$$

What is $g_2'(x)$?

By rules of Differentiation we get

$$g_2'(x) = \begin{cases} -\cos(1/x) + 2x \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

g_2' exists everywhere on $\mathbb{R}$ but g_2' is not continuous

($\lim_{x\to 0} g_2'(x)$ does not exist)

MATH 400

Real Analysis

Video #28

Applications of Differential Calculus are based on finding max/min values of $f(x)$ by ~~solving~~ $f'(x)=0$.

Theorem [Interior Extremum Theorem / Fermat's Thm]

Let f be differentiable on (a,b).

If f attains a max value at $c \in (a,b)$
(i.e., $f(c) \geq f(x)\ \forall x \in (a,b)$)

then $f'(c)=0$

Same is true for $f(c)$ is min value.

Proof $[f(c) \geq f(x)\ \forall x \in (a,b) \Rightarrow f'(c) = 0]$

Since $c \in (a,b)$, an open interval, we can find $(x_n), (y_n) \subseteq (a,b)$ s.t. $c - \frac{1}{n} < x_n < c$ and $c < y_n < c + \frac{1}{n}$ $\forall n$.

Since $f(c)$ is a maximum, $f(y_n) - f(c) \leq 0$ $\forall n$

and $y_n - c \geq 0$ $\forall n$

which implies $\boxed{\dfrac{f(y_n) - f(c)}{y_n - c} \leq 0}$ $\forall n$

By Order Limit Thm,

$$f'(c) = \lim_{n\to\infty} \frac{f(y_n) - f(c)}{y_n - c} \leq 0$$

since $y_n \to c$

Proof $[f(c) \geq f(x)\ \forall x \in (a,b) \Rightarrow f'(c) = 0]$

Since $c \in (a,b)$, an open interval, we can find $(x_n), (y_n) \subseteq (a,b)$ s.t. $c - \frac{1}{n} < x_n < c$ and $c < y_n < c + \frac{1}{n}$ $\forall n$.

Since $f(c)$ is a maximum, $f(y_n) - f(c) \leq 0\ \forall n$

and $y_n - c \geq 0\ \forall n$

which implies $\dfrac{f(y_n) - f(c)}{y_n - c} \leq 0\ \forall n$

By Order Limit Thm, $f'(c) = \lim\limits_{n \to \infty} \dfrac{f(y_n) - f(c)}{y_n - c} \leq 0$ — ①

Also $f(x_n) - f(c) \leq 0\ \forall n$ and $x_n - c \leq 0\ \forall n$

which implies $\dfrac{f(x_n) - f(c)}{x_n - c} \geq 0\ \forall n$

Again by OL Thm, $f'(c) = \lim \dfrac{f(x_n) - f(c)}{x_n - c} \geq 0$ — ②

① & ② $\Rightarrow f'(c) = 0$

<u>We know</u>: f differentiable $\Rightarrow$ f continuous $\Rightarrow$ f has IVP

Intermediate Value Property

What about the function f'?

We know f' exists $\not\Rightarrow$ $(f')'$ exists

We know f' exists $\not\Rightarrow$ f' continuous

Does f' have the IVP??

We know: f differentiable $\Rightarrow f$ continuous $\Rightarrow f$ has IVP

Intermediate Value Property

What about the function f'?

we know f' exists $\not\Rightarrow (f')'$ exists

we know f' exists $\not\Rightarrow f'$ continuous

Theorem [Darboux' Theorem]

Let f be differentiable on $[a,b]$.

If α satisfies $f'(a)<\alpha<f'(b)$ (or $f'(b)<\alpha<f'(a)$)

then $\exists\, c \in (a,b)$ s.t. $f'(c)=\alpha$.

We know: f differentiable $\Rightarrow$ f continuous $\Rightarrow$ f has IVP

Intermediate Value Property

What about the function f'?

We know f' exists $\not\Rightarrow$ $(f')'$ exists

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Theorem [Darboux' Theorem]

Let f be differentiable on $[a,b]$.

If α satisfies $f'(a) < \alpha < f'(b)$ (or $f'(b) < \alpha < f'(a)$)

then $\exists\, c \in (a,b)$ s.t. $f'(c) = \alpha$.

Proof: WLOG assume $f'(a) < \alpha < f'(b)$

Define $\boxed{g(x) = f(x) - \alpha x}$

Then g is differentiable on $[a,b]$ & $g'(x) = f'(x) - \alpha$

Prop $\left[f'(a) < \alpha < f'(b) \Rightarrow \exists\, c \in (a,b) \text{ s.t. } f'(c) = \alpha \right]$

$g(x) = f(x) - \alpha x$ & $g'(x) = f'(x) - \alpha$ (why?)

$g'(a) = f'(a) - \alpha < 0$ $g'(b) = f'(b) - \alpha > 0$

We want $c \in (a,b)$ s.t. $g'(c) = 0$ i.e., $f'(c) = \alpha$

Step 1a [HW] $\exists\, c_1 \in (a,b)$ s.t. $g(c_1) < g(a)$ (i.e. $g(a)$ is not a min)

Step 1b [HW] $\exists\, c_2 \in (a,b)$ s.t. $g(c_2) < g(b)$ (i.e. $g(b)$ is not a min)

Step 2 [HW] $\exists\, c \in (a,b)$ s.t. $g'(c) = 0$

Prop [$f'(a) < \alpha < f'(b) \Rightarrow \exists c \in (a,b)$ s.t. $f'(c) = \alpha$]

$g(x) = f(x) - \alpha x$ & $g'(x) = f'(x) - \alpha$ (why?)

$g'(a) = f'(a) - \alpha < 0$ $g'(b) = f'(b) - \alpha > 0$

We want $c \in (a,b)$ s.t. $g'(c) = 0$ i.e., $f'(c) = \alpha$

Step 1a [HW] $\exists c_1 \in (a,b)$ s.t. $g(c_1) < g(a)$

$g'(a) = \lim_{n\to\infty} \frac{g(x_n) - g(a)}{x_n - a} < 0$ choosing $x_n > a$ means numerator must become negative

Step 1b [HW] $\exists c_2 \in (a,b)$ s.t. $g(c_2) < g(b)$

— same idea a step 1a. —

Step 2 [HW] $\exists c \in (a,b)$ s.t. $g'(c) = 0$

what does Extreme Value Thm tell us about g in $[a,b]$?

what does Interior Extremum Thm tell us about g' in (a,b)?