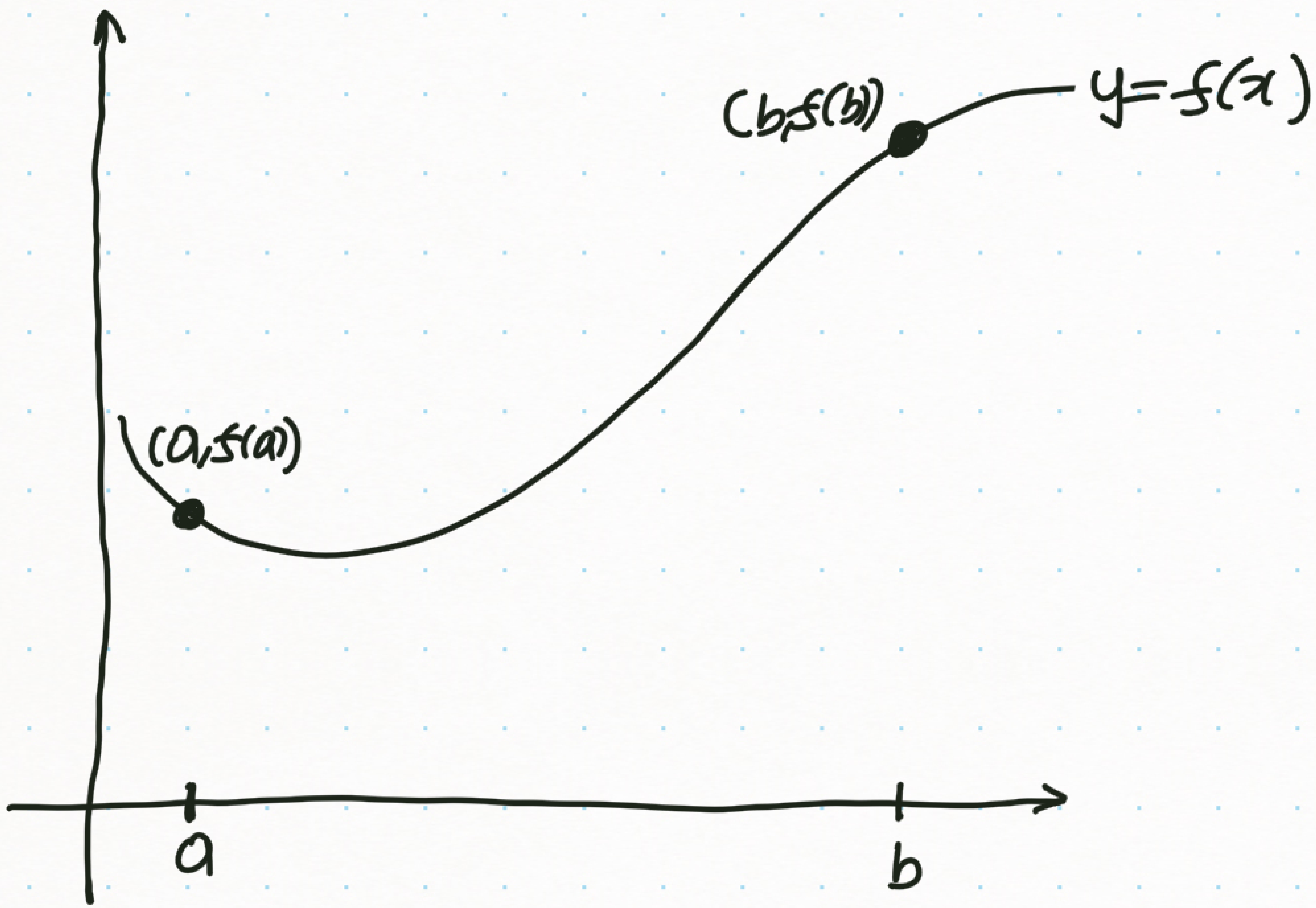
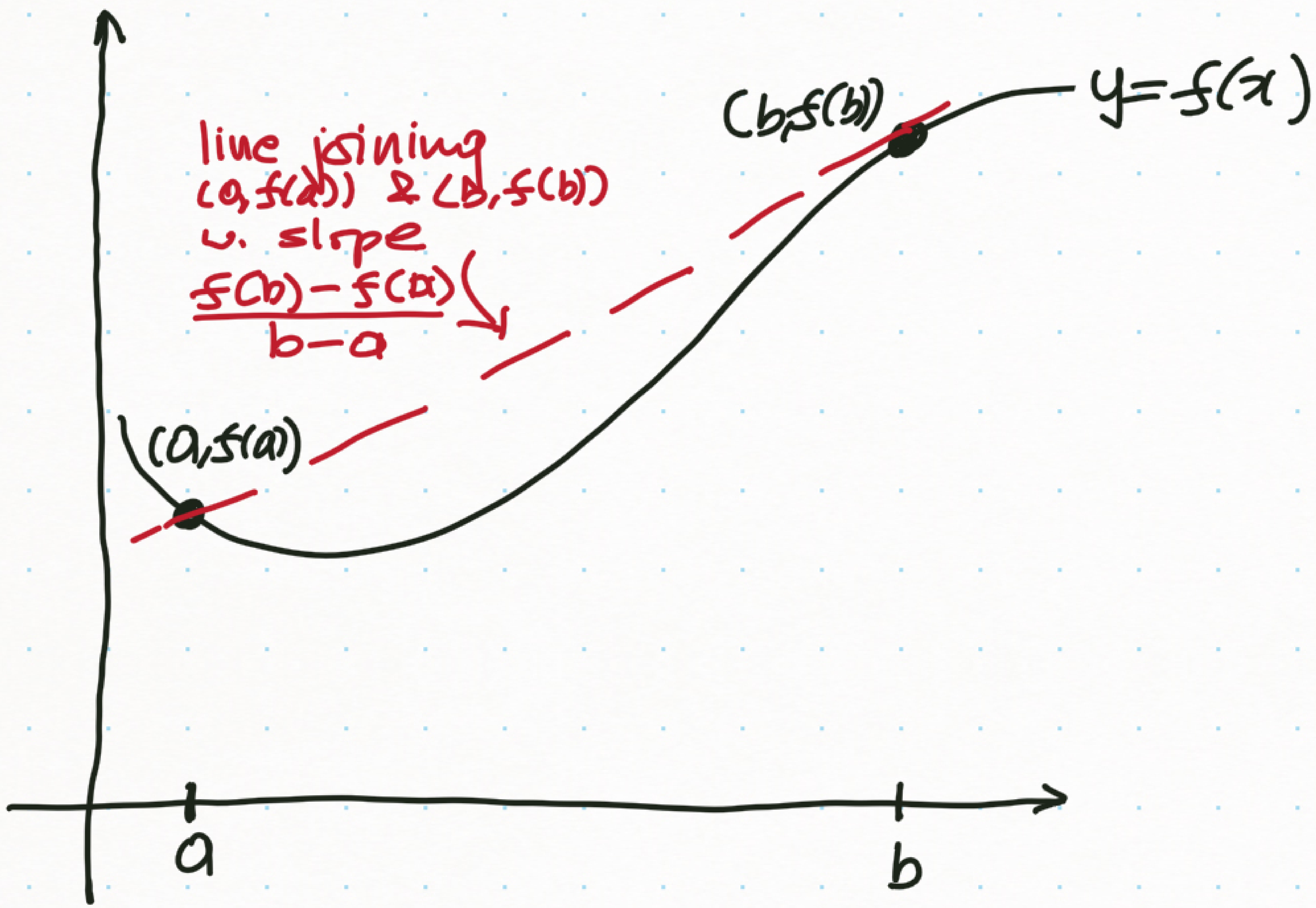
Math 400

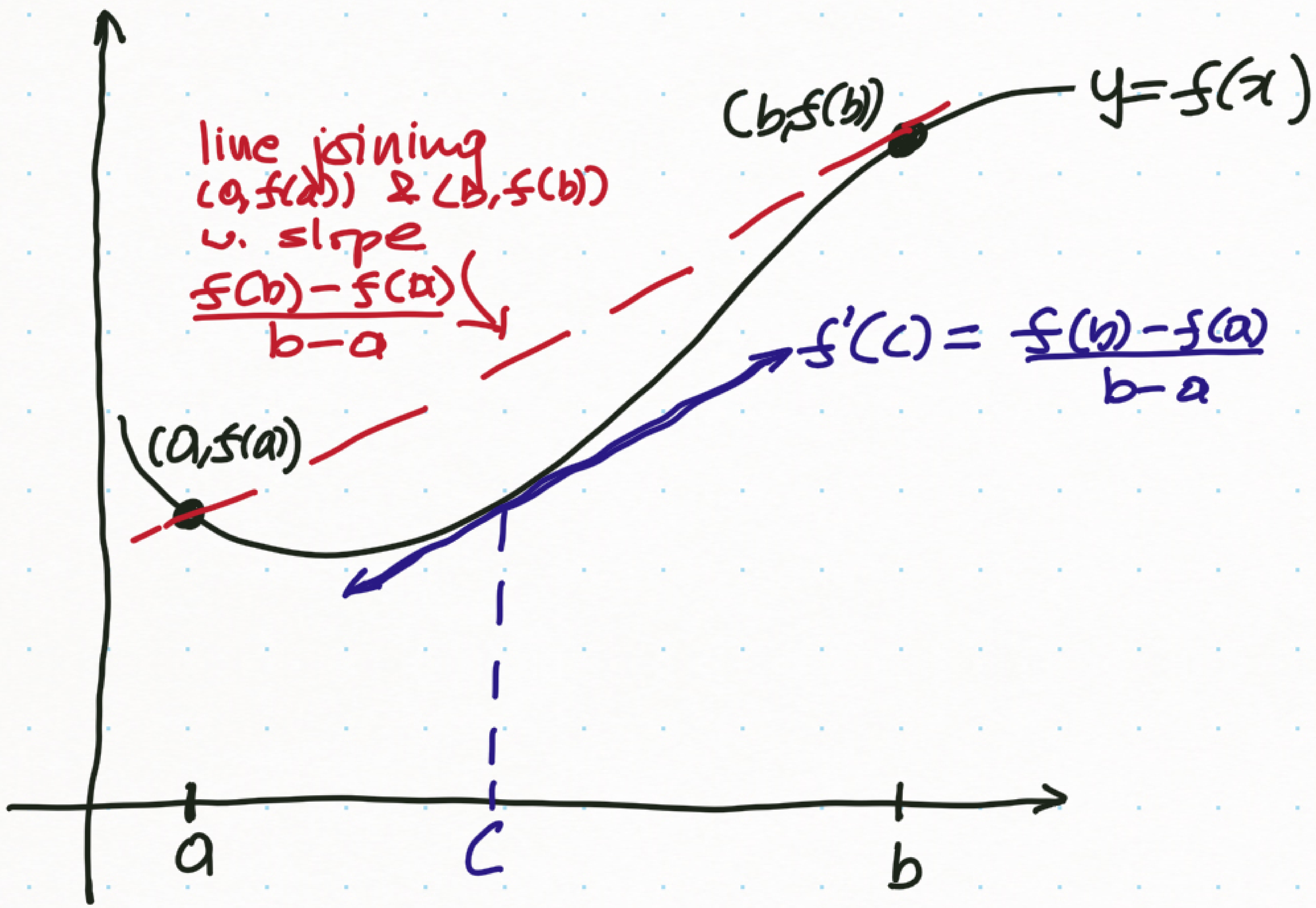
Real Analysis

Video #29



$y=f(x)$
$(b,f(b))$
$(a,f(a))$
a
b

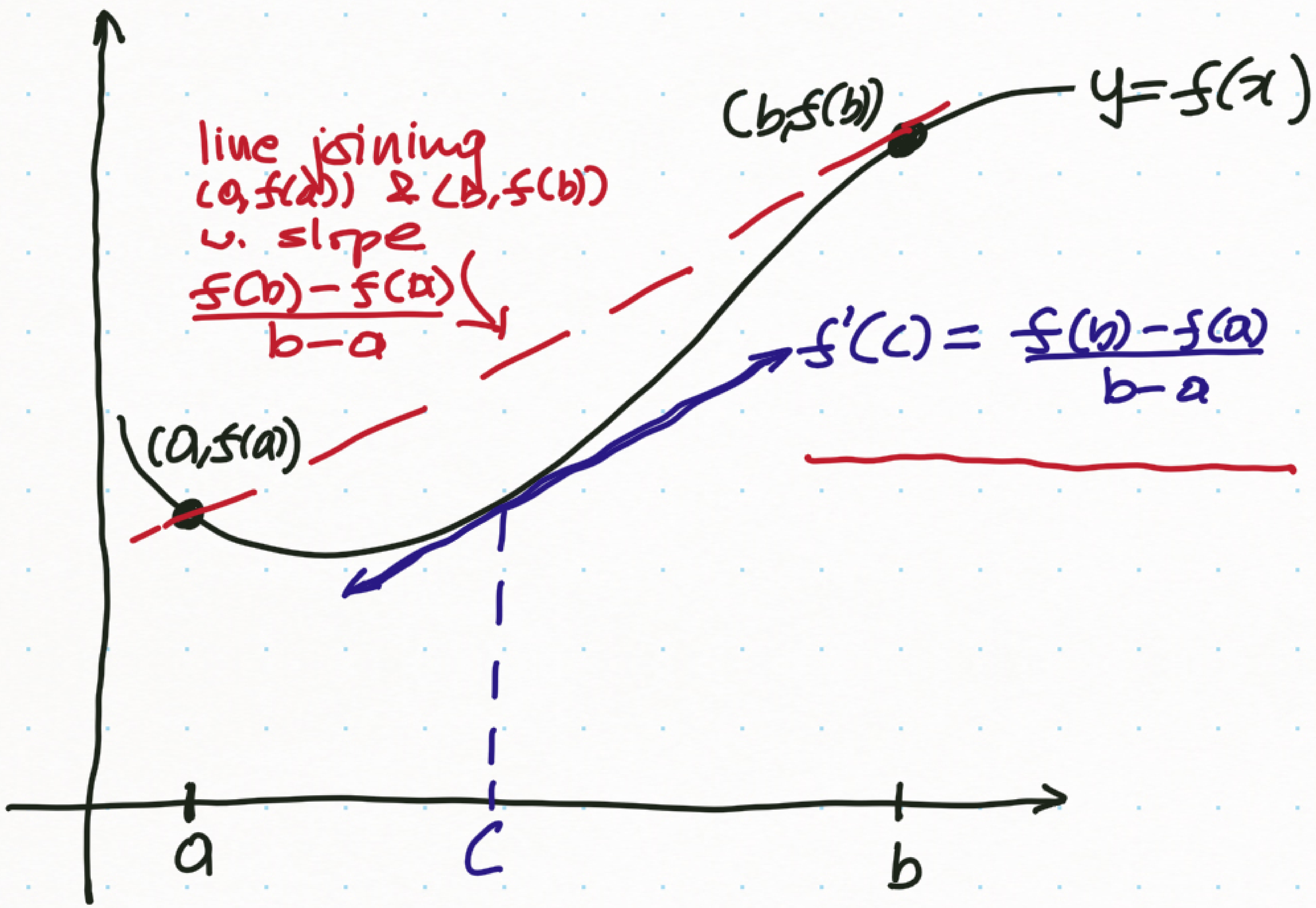

(b,f(b))
y=f(x)
line joining
(a,f(a)) & (b,f(b))
w. slope
$\frac{f(b)-f(a)}{b-a}$
(a,f(a))
a
b



Mean Value Theorem

$\exists\, c \in (a,b)$ s.t. $f'(c) = \frac{f(b)-f(a)}{b-a}$

A sort of IVT for derivatives



We have already done the hard work.

f on $[a,b]$ achieves it max & min by EVT

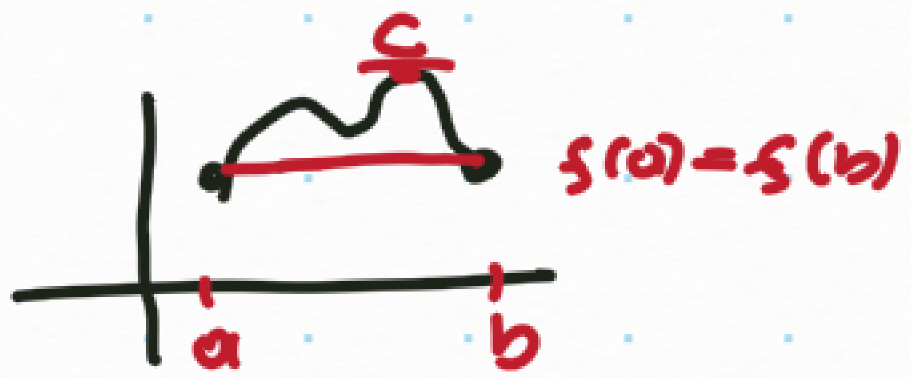
combine with Interior Extremum Thm. that $f'(c)=0$ when c is max/min

Mean Value Theorem

$\exists\, c \in (a,b)$ s.t. $f'(c)=\frac{f(b)-f(a)}{b-a}$

Rolle's Theorem Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b).
If $f(a)=f(b)$ then $\exists\, c\in(a,b)$ s.t. $f'(c)=0$

Note $f'(c)=\dfrac{f(b)-f(a)}{b-a}=0$



Proof f is cont. on a compact set, so f attains its max & min
If max & min occur on a or b then f is a constant function
and $f'(c)=0\ \forall\, c\in(a,b)$

If max or min occur on $c\in(a,b)$ then by IFT $f'(c)=0$.

$\blacksquare$

Mean Value Theorem

If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c\in(a,b)$ s.t.

$$f'(c)=\frac{f(b)-f(a)}{b-c}$$

Proof (Idea: Reduce to Rolle's Thm)

The equation of the line through $(a,f(a))$ and $(b,f(b))$ is

$$y=\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)+f(a)$$

Consider the difference between this line and $y=f(x)$:

$$d(x)=f(x)-\left[\left(\frac{f(b)-f(a)}{b-a}\right)(x-a)+f(a)\right]$$

d is continuous on $[a,b]$ & differentiable on (a,b) [By Algebra of cont. & diff. function

Mean Value Theorem

If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c\in(a,b)$ s.t.

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d is continuous on $[a,b]$ & differentiable on (a,b) [By Algebra of cont. & diff. function]

and $d(a)=0=d(b)$

By Rolle's Thm applied to d, $\exists\, c\in(a,b)$ s.t. $d'(c)=0$.

$$d'(c)=f'(c)-\frac{f(b)-f(a)}{b-a},$$ i.e., $\exists\, c$ s.t. $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

Cor If $g: I \to \mathbb{R}$ is differentiable on an interval I and $g'(x) = 0 \ \forall x \in I$, then $g(x) = k$ for some constant k

Proof Let $x_1, x_2 \in I$ with $x_1 < x_2$. We want to show $g(x_1) = g(x_2)$.

By MVT applied to g on $[x_1, x_2]$:

$\exists \ c \in (x_1, x_2) \subseteq I$ s.t. $g'(c) = \dfrac{g(x_2) - g(x_1)}{x_2 - x_1}$

Since $g'(c) = 0$, we get $g(x_1) = g(x_2)$.

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Since $g'(c) = 0$, we get $g(x_1) = g(x_2)$.

Cor If f and g are differentiable functions on an interval I and satisfy $f'(x) = g'(x)$ $\forall x \in I$, then $f(x) = g(x) + k$ for some constant k.

Proof Discussion Ques.

Recall, f increasing means $f(x_1) \leq f(x_2)$ for any $x_1 < x_2$.

Cor Let $f: I \to \mathbb{R}$ be differentiable on the interval I.

(i) f is increasing $\iff f'(x) \geq 0 \ \forall x \in I$

(ii) f is decreasing $\iff f'(x) \leq 0 \ \forall x \in I$

Proof Assume f is increasing.

This means $f(x) - f(c)$ and $x - c$ for any $x, c \in I$ are either both nonnegative (≥ 0) or both nonpositive (≤ 0)

$\therefore$ for any $x \neq c$, $\boxed{\dfrac{f(x)-f(c)}{x-c}} \geq 0 \quad \forall x, c \in I$

Hence $f'(c) = \lim\limits_{x \to c} \dfrac{f(x)-f(c)}{x-c} \geq 0$ as needed.

Recall, f increasing means $f(x_1) \leq f(x_2)$ for any $x_1 < x_2$.

Cor Let $f: I \to \mathbb{R}$ be differentiable on the interval I.

(i) f is increasing $\iff$ $f'(x) \geq 0 \ \forall x \in I$

(ii) f is decreasing $\iff$ $f'(x) \leq 0 \ \forall x \in I$

Proof Assume $f'(x) \geq 0 \ \forall x \in I$

For any $x_1, x_2 \in I$ with $x_1 < x_2$,

by MVT $\exists\, c \in (x_1, x_2) \subseteq I$ s.t. $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

That is, $f(x_2) - f(x_1) = f'(c)\,(x_2 - x_1)$

≥ 0 since $x_2 \geq x_1$

and $f'(c) \geq 0$

ie, $f(x_2) \geq f(x_1)$.

Generalized Mean Value Theorem

If f and g are continuous on $[a,b]$ and differen. on (a,b) then $\exists\, c \in (a,b)$ s.t. $[f(b)-f(a)]\,g'(c) = [g(b)-g(a)]\,f'(c)$

If g' is never zero on (a,b) then we can say

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof Apply MVT to $h(x) = [f(b)-f(a)]\,g(x) - [g(b)-g(a)]\,f(x)$

L'Hospital's Rules for evaluating $\lim\limits_{x \to c} \frac{f(x)}{g(x)}$

$$= \frac{\lim f(x)}{\lim g(x)}$$ (assuming both limits exist)

Theorem [L'Hospital for 0/0 form]

Let I be an open interval containing pt. a.
Suppose f and g are differentiable on I, except possibly a.

If $f(a) = g(a) = 0$, and $g'(x) \neq 0 \ \forall x \neq a$,

then $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = L \implies \lim\limits_{x \to a} \frac{f(x)}{g(x)} = L$

Proof [HW?]
- Write the definition of $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = L$ to find a δ
- Apply GMVT to f and g in $[a, x]$ (& $[x, a]$)
- Verify the definition of $\lim\limits_{x \to a} \frac{f'(x)}{g'(x)} = L$

Definition $\left[\lim_{x\to c} f(x) = \infty\right]$

Given $f: I \to \mathbb{R}$ and a limit point c of I, we say $\lim_{x\to c} f(x) = \infty$ if $\forall M > 0$ $\exists \delta > 0$ s.t.

$$0 < |x-c| < \delta \Rightarrow f(x) \geq M.$$

Similarly, define $\lim f(x) = -\infty$

Definition $[\lim_{x\to c} f(x)=\infty]$

Given $f: I \to \mathbb{R}$ and a limit point c of I, we say $\lim_{x\to c} f(x) = \infty$ if $\forall M>0\ \exists \delta>0$ s.t.

$$0<|x-c|<\delta \Rightarrow f(x) \geq M.$$

Theorem [L'Hospital for ∞/∞ form]

Suppose f and g are differentiable on (a,b) and $g'(x) \neq 0\ \forall x \in (a,b)$.

If $\lim_{x\to a} g(x) = \infty$ (or $-\infty$), then

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L \Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = L$$

MATH 400

Real Analysis

Video # 30

We want understand convergence of

— sequence of functions

— series of functions

and build towards justifying Taylor / Mclaurin series expansions of functions.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Let $f_n : A \to \mathbb{R}$ be a function for each $n \in \mathbb{N}$

<u>Defn</u> The sequence (f_n) of functions <u>converges pointwise</u> on A to a function f if for all $x \in A$, the sequence of numbers $f_n(x)$ converges to $f(x)$ as $n \to \infty$.

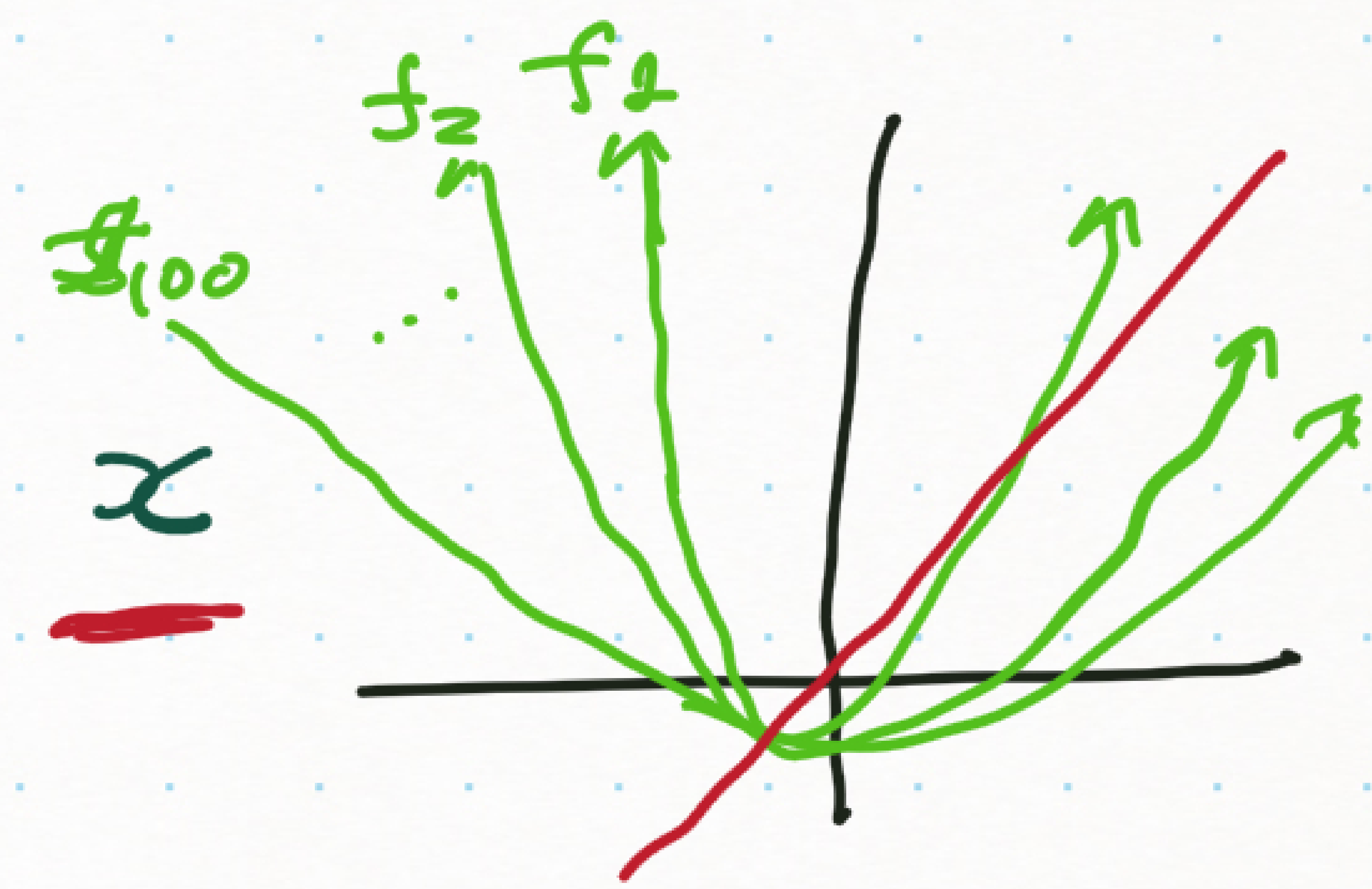
we write $\lim_{n\to\infty} f_n(x) = f(x)$ or $f_n \to f$

<u>Examples</u>

① $f_n(x) = (x^2 + nx)/n$ on $\mathbb{R}$

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2+nx}{n} = \lim_{n\to\infty} \left(\frac{x^2}{n} + x\right) = \underline{x}$$

$\therefore f_n \to f$ pointwise where $f(x) = x$



If f_n is continuous for each n, Then is f (where $f_n \to f$) also continuous?

Let $f_n \to f$ where each f_n is continuous.

To show: f is continuous we have to show $|f(x)-f(c)| < \epsilon$ when $|x-c| < \delta$

$$|f(x)-f(c)| = |f(x)-f_k(x)+f_k(x)-f_k(c)+f_k(c)-f(c)|$$

$$\leq \underbrace{|f_k(x)-f(x)|}_{<\epsilon/3} + \underbrace{|f_k(x)-f_k(c)|}_{<\epsilon/3} + \underbrace{|f_k(c)-f(c)|}_{<\epsilon/3}$$

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since $f_k \to f$ for $k \geq k_0$

If f_n is continuous for each n, Then is f (where $f_n \to f$) also continuous?

Let $f_n \to f$ where each f_n is continuous.

To show: f is continuous we have to show $|f(x)-f(c)| < \epsilon$ when $|x-c| < \delta$

$$|f(x)-f(c)| = |f(x)-f_k(x)+f_k(x)-f_k(c)+f_k(c)-f(c)|$$

$$\leq \underbrace{|f_k(x)-f(x)|}_{< \epsilon/3} + \underbrace{|f_k(x)-f_k(c)|}_{< \epsilon/3 \text{ since } f_k \text{ is continuous}} + \underbrace{|f_k(c)-f(c)|}_{< \epsilon/3 \text{ since } f_k \to f}$$

?? $\leftarrow$ for $k \geq k_0$ $\leftarrow$ for $k \geq k_0$

same k_0 may not work

Examples

(2) Let $g_n(x) = x^n$ on $[0,1]$.

We know $x^n \to 0$ if $x \in [0,1)$ & $x^n \to 1$ if $x = 1$

$\therefore g_n \to g$ pointwise where $g(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

Each g_n is continuous but g is not.

Examples

(2) Let $g_n(x) = x^n$ on $[0,1]$.

We know $x^n \to 0$ if $x \in [0,1)$ & $x^n \to 1$ if $x = 1$

$\therefore$ $g_n \to g$ pointwise where $g(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

Each g_n is continuous but g is not.

(3) Let $h_n(x) = x^{1+\frac{1}{2n-1}}$ on $[-1,1]$

For each fixed $x \in [-1,1]$

$$\lim_{n\to\infty} h_n(x) = x \overbrace{\lim_{n\to\infty} x^{\frac{1}{2n-1}}}^{\substack{-1 \text{ if } x<0 \\ = +1 \text{ if } x>0}} = |x| = h(x)$$

Each h_n is differentiable but h is not.

ϵ-N definition of pointwise convergence of f_n

$f_n \to f$ means For x, $\forall \epsilon > 0$ $\exists N$ (possibly dependent on x) s.t.

$$|f_n(x) - f(x)| < \epsilon \quad \forall n \geq N.$$

we want N to work for all x simultaneously.

Definition $f_n : A \to \mathbb{R}$ be a sequence of functions (f_n) converges uniformly on A to f defined on A if $\forall \epsilon > 0$ $\exists N$ s.t. $|f_n(x) - f(x)| < \epsilon$ for all $n \geq N$ and $x \in A$

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Examples ① $g_n(x) = \dfrac{1}{n(1+x^2)}$ on $\mathbb{R}$.

For any fixed $x \in \mathbb{R}$, $g_n(x) \to 0$ as $n \to \infty$, so $g(x) = 0$ is the pointwise limit.

Uniform? $|g_n(x) - g(x)| = \left|\dfrac{1}{n(1+x^2)} - 0\right| = \left|\dfrac{1}{n(1+x^2)}\right| \leq \dfrac{1}{n} \quad \forall x \in \mathbb{R}$

Given $\epsilon > 0$, $\exists N > 1/\epsilon$ s.t. $|g_n(x) - g(x)| < \epsilon$ for $n \geq N$

② $f_n(x) = (x^2+nx)/n \longrightarrow f(x) = x$ pointwise

On $\mathbb{R}$, this convergence is not uniform

$|f_n(x) - f(x)| = \left|\frac{x^2+nx}{n} - x\right| = \frac{x^2}{n} < \epsilon$ means $N > \frac{x^2}{\epsilon}$

On $[-b, b]$, this convergence is uniform

$|f_n(x) - f(x)| = \frac{x^2}{n} \leq \frac{b^2}{n} < \epsilon$ means $N > \frac{b^2}{\epsilon}$

Theorem [Cauchy Criterion for Uniform Convergence]

(f_n) seq. defined on $A \subseteq \mathbb{R}$ converges uniformly on A

iff $\underbrace{\forall \epsilon > 0 \ \exists N \text{ s.t. } |f_n(x) - f_m(x)| < \epsilon \ \forall m, n > N \text{ and } x \in A}_{(f_n) \text{ is Cauchy}}$

Proof

Using Cauchy criterion for convergence of sequence (of numbers)

Continuous Limit Theorem

Let (f_n) converge uniformly on A to f.
If each f_n is continuous at $c \in A$ then f is continuous at c.

Proof Fix $c \in A$ & let $\epsilon > 0$.

By Unif. Convg. Choose N s.t. $|f_k(x) - f(x)| < \epsilon/3 \quad \forall k \geq N \;\&\; x \in A$

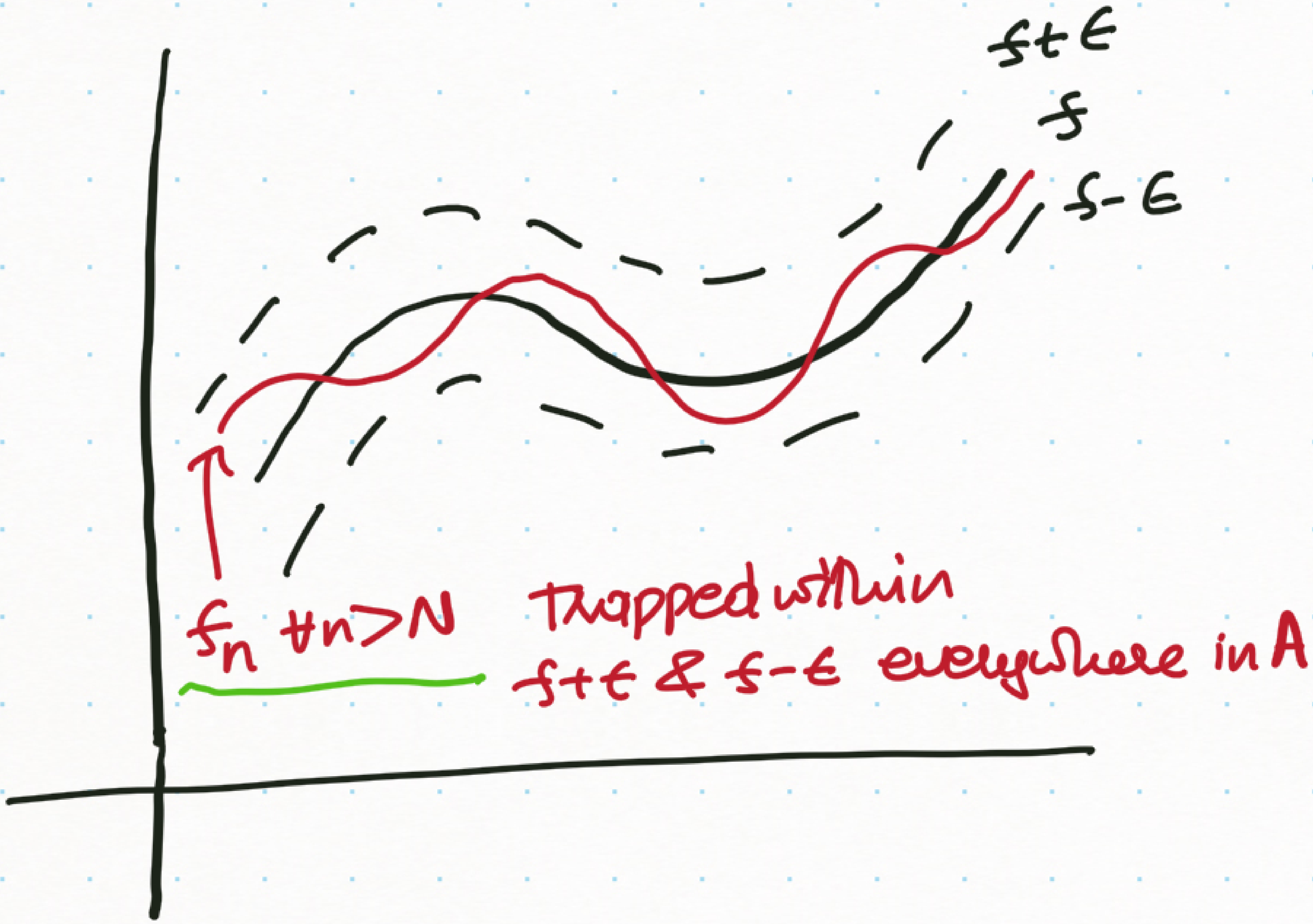
so $|f_N(x) - f(x)| < \epsilon/3 \quad \forall x \in A.$ $(k = N)$

By f_N continuous $\exists\, \delta > 0$ s.t. $|f_N(x) - f_N(c)| < \epsilon/3$ for $|x - c| < \delta$

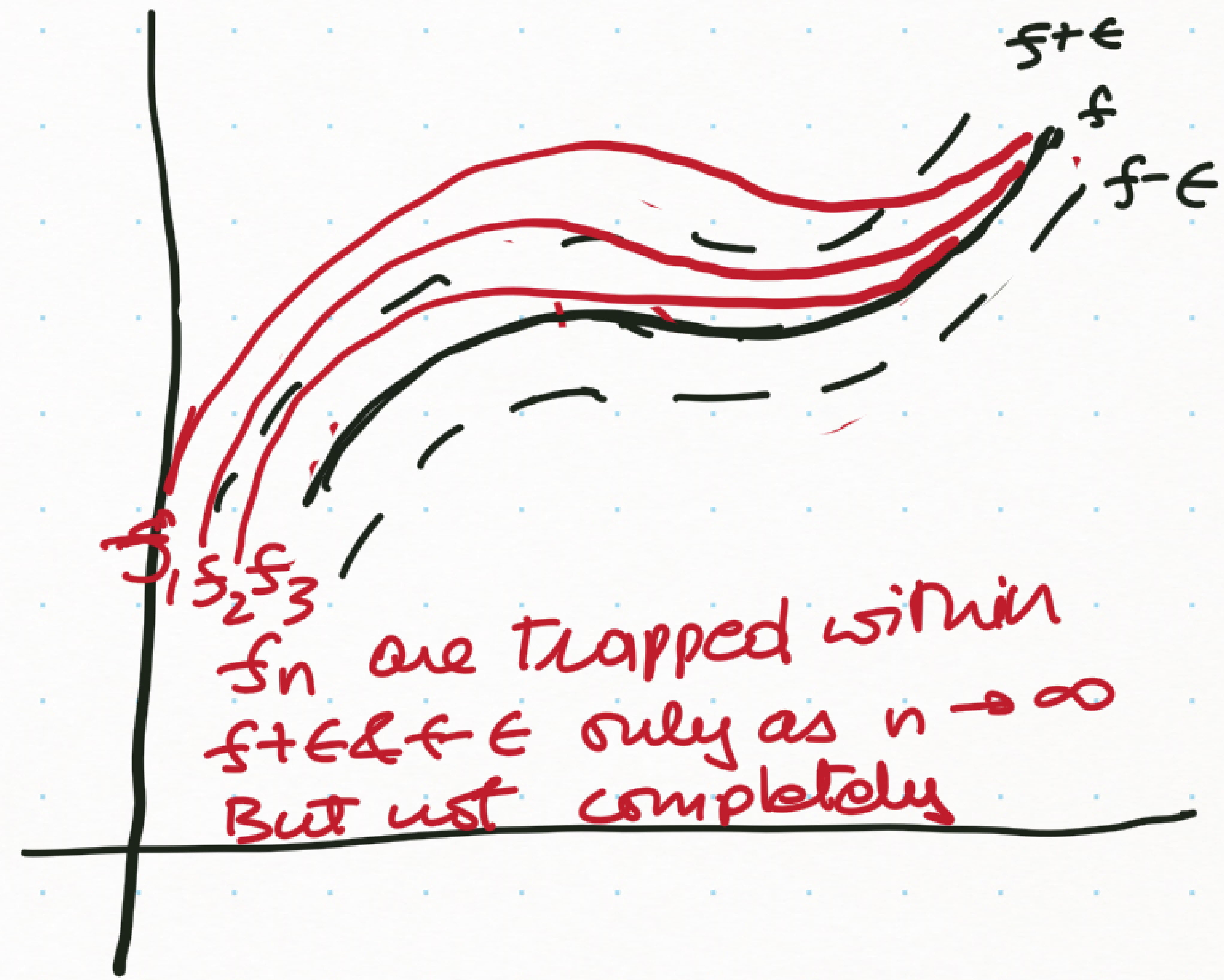
$$\therefore |f(x) - f(c)| = |f(x) - f_N(x) + f_N(x) - f_N(c) + f_N(c) - f(c)|$$

$$\leq |f(x) - f_N(x)| + |f_N(x) - f_N(c)| + |f_N(c) - f(c)|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$



Uniform Convergence



Pointwise convergence

We already saw pointwise convergence does not preserve differentiability.
Does uniform convergence preserve differentiability?

<u>Example</u> $f_n : [-1,1] \to \mathbb{R}$ with $f_n(x) = x^{1+1/(2n-1)}$

we saw $(f_n) \longrightarrow f(x) = |x|$ pointwise

Also check (!) (f_n) uniformly converges to f

Each f_n is differentiable on $[-1,1]$

but f is not differentiable at 0.

We already saw pt.wise/Unif. convergence does not preserve differentiability.
Does uniform convergence "preserve derivatives" ?

Example let $g_n : [-2,2] \to \mathbb{R}$ as $g_n(x) = \dfrac{x}{1+nx^2}$

(g_n) converges to $g(x) = 0$ both pointwise and uniformly.

However, note $g_n'(0) = 1 \ \forall n$ $\left(g_n'(x) = \dfrac{1-nx^2}{(1+kx^2)^2}, \text{ by Q. Rule} \right)$

But $g'(0) = 0$

So the derivatives may not match.

Differentiability Limit Theorem

Let $f_n \to f$ pointwise on $[a,b]$, and assume f_n' exists for all n.

If (f_n') converges uniformly on $[a,b]$ to g then f is differentiable and $f' = g$.

we need uniform convergence of (f_n') to ensure that limit of (f_n) preserves differentiability and the derivatives match.

Differentiability Limit Theorem (Stronger)

Let ~~$f_n \to f$ pointwise on $[a,b]$~~, and assume f_n' exists for all n.

If (f_n') converges uniformly on $[a,b]$ to g

then f is differentiable and $f' = g$.

This can be replaced by a weaker requirement:

$\exists\, x_0 \in [a,b]$ s.t. $\underbrace{f_n(x_0)}_{\text{sequence of numbers}}$ is convergent

This gives us: $f_n \to f$ uniform convergence. & $f' = g$

Math 400

Real Analysis

Video #31

Please review "Series of numbers".

<u>Defn</u> Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$.
let $S_k(x) = f_1(x) + f_2(x) + \cdots + f_k(x)$ be seq. of partial sums.

① The series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to $f: A \to \mathbb{R}$ if $S_k(x)$ converges pointwise to $f(x)$.

② The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A to $f: A \to \mathbb{R}$ if $S_k(x)$ converges uniformly on A to $f(x)$.

Everything we want to understand about
$\sum_{n=1}^{\infty} f_n(x) = f(x)$ reduces to understand how
the sequence of partial sums $(S_k(x))$ behaves under
the mode of convergence (pointwise or uniform).

Theorem Let $f_n : A \to \mathbb{R}$ be continuous $\forall n$.
Assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to a function f.
Then, f is continuous on A.

Everything we want to understand about $\sum_{n=1}^{\infty} f_n(x) = f(x)$ reduces to understand how the sequence of partial sums $(S_k(x))$ behaves under the mode of convergence (pointwise or uniform).

<u>Theorem</u> Let $f_n : A \to \mathbb{R}$ be continuous $\forall n$.
Assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to a function f.
Then, f is continuous on A.

<u>Proof</u> f_n continuous functions $\Rightarrow S_k = \sum_{i=1}^{k} f_i$ is continuous (Why?)
By Continuous limit Thm., f is continuous since $S_k(x) \to f(x)$ uniformly on A.

Theorem Let $f_n : I \to \mathbb{R}$ be differentiable on interval I $\forall n$.
Assume $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to $g(x)$ on I.
If $\exists\, x_0 \in I$ s.t. $\sum_{n=1}^{\infty} f_n(x_0)$ converges, then

- $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to $f(x)$ on I

and $f(x)$ is differentiable with $f'(x) = g(x)$ on I.

i.e., $f(x) = \sum_{n=1}^{\infty} f_n(x)$ and $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$

Proof follows directly from Differentiable Limit Theorem (Stronger) applied to $S_k(x)$.

Theorem [Cauchy Criterion for Unif. Convergence of Series]

Series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $A \subseteq \mathbb{R}$

$\Longleftrightarrow$ $\forall \epsilon > 0$, $\exists$ N s.t. $|f_{m+1}(x) + \dots + f_n(x)| < \epsilon$ $\forall n > m \geq N$ and $x \in A$.

Theorem [Cauchy Criterion for Unif. Convergence of Series]

Series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $A \subseteq \mathbb{R}$

$\iff \forall \epsilon > 0, \exists N$ s.t. $|f_{m+1}(x) + \dots + f_n(x)| < \epsilon \ \forall n > m \geq N$ and $x \in A$.

Cor [Weierstrass M-Test]

For each n, let $f_n : A \to \mathbb{R}$ and $M_n \in \mathbb{R}^+$ s.t. $|f_n(x)| \leq M_n \ \forall x \in A$.

If $\sum_{n=1}^{\infty} M_n$ converges then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A.

Proof [Fill in details]

$\sum_{n=1}^{\infty} M_n$ converges $\Rightarrow$ partial sum of (M_n) are Cauchy $\Rightarrow$ $\sum f_n(x)$ is Cauchy $\Rightarrow$ $\sum f_n(x)$ converges unif.

<u>Example</u> Let $f_n(x) = \dfrac{1}{x^4 + 3x^2 n + n^2 + 7}$ on $\mathbb{R}$

What is behavior of $\sum_{n=1}^{\infty} f_n(x)$?

<u>Example</u> Let $f_n(x) = \dfrac{1}{x^4 + 3x^2 n + n^2 + 7}$ on $\mathbb{R}$

What is behavior of $\sum_{n=1}^{\infty} f_n(x)$?

$$|f_n(x)| = \frac{1}{x^4 + 3x^2 n + n^2 + 7} \leq \frac{1}{n^2}$$

(since every term in denominator is nonnegative)

Let $M_n = \dfrac{1}{n^2}$ Then $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges by series p-test $(p=2)$

$\therefore$ by Weierstrass M-Test, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $\mathbb{R}$.

Power series is a function of the form $\sum_{n=0}^{\infty} a_n x^n$

For which values of x does a power series converge?

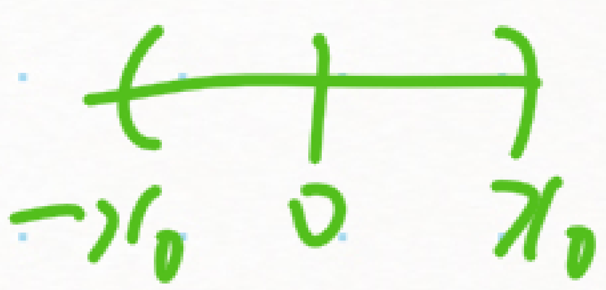
e.g. $\sum_{k=0}^{\infty} x^k$ is a power series with each $a_k = 1$.

It is a geometric series with common ratio $r = x$, that converges precisely when $r \in (-1, 1)$

$\therefore \sum_{k=0}^{\infty} x^k$ converges to $\frac{1}{1-x}$ when $x \in (-1, 1)$.

Theorem If $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for each x s.t. $|x| < |x_0|$.

$x \in (-|x_0|, |x_0|)$



Proof Suppose $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then the sequence of terms $(a_n x_0^n)$ converges to 0

so, $(a_n x_0^n)$ is bounded.

That is, $\exists M > 0$ s.t. $|a_n x_0^n| \le M \quad \forall n$

If $x \in \mathbb{R}$ satisfies $|x| < |x_0|$, Then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \le M \left|\frac{x}{x_0}\right|^n \quad \forall n$$

Note $\sum_{n=0}^{\infty} M\left|\frac{x}{x_0}\right|^n$ is a geometric series with $r = \left|\frac{x}{x_0}\right| < 1$ & converges.

By series comparison test, $\sum |a_n x^n|$ converges.

Theorem If $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for each x s.t. $|x| < |x_0|$.

$\hookrightarrow$ $x \in (-|x_0|, |x_0|)$, an interval of values where $\sum a_n x^n$ converges

So, convergence of $\sum a_n x^n$ must occur on an interval

i.e., $\underset{\underset{R=0}{[0,0]}}{\{0\}}$, $\underset{\underset{R=\infty}{(-\infty,\infty)}}{\mathbb{R}}$, or $(-R,R)$ or $[-R,R)$ or $(-R,R]$ or $[-R,R]$.

The value R above is called <u>Radius of Convergence</u> of a power series.

e.g. $\sum x^n$ has $R=1$ (since its convergent on $(-1,1)$)

Theorem [Uniform Convergence of Power Series]

If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at $x_0 \in \mathbb{R}$

then $\sum a_n x^n$ converges uniformly on $[-c, c]$

where $c = |x_0|$.

Proof Apply Weierstrass M-test

$\sum a_n x^n$ converges absolutely at x_0 means $\sum_{n=0}^{\infty} |a_n x_0^n|$ converges

Let $M_n = |a_n x_0^n|$ then $\sum M_n$ converges.

If $x \in [-c, c]$ then $|a_n x^n| \leq |a_n c^n| = |a_n x_0^n| = M_n \quad \forall n$

So by Weierstrass M-test, $\sum a_n x^n$ converges uniformly on $[-c, c]$.

Math 400

Real Analysis

Video #32

Recall, we know

Thm If power series $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$ then it converges absolutely for every $x \in (-|x_0|, |x_0|)$ ⟸ Interval of convergence

Thm If power series $\sum a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on $[-|x_0|, |x_0|]$

Since each $a_n x^n$ is cont. & $\sum a_n x^n$ converges uniformly we get $\sum a_n x^n$ is also continuous..

Are power series differentiable? Is term-by-term differentiation allowed?

Yes, but proofs will be tedious unless we develop some more theory.

<u>Theorem</u> Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on an interval $I \subseteq \mathbb{R}$. Then,

f is continuous on I and differentiable on any $(-R, R) \subseteq I$.

The derivative is given by $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.

Moreover, f is infinitely differentiable on $(-R, R)$ and the successive derivatives are obtained by term-by-term differentiation of the previous series.

What about "Integrability"?

$\hookrightarrow$ Antiderivative

Theorem Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$

Then $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ is defined on $(-R, R)$

and $F'(x) = f(x)$ (That is, we can do term-by-term antidifferentiation "Integration")

We can use the previous two tools to create new power series from known ones.

e.g. we know $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x \in \underbrace{(-1,1)}_{\text{interval of convergence}}$

i.e. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots$ for $x \in (-1,1)$

Then, we get

① $\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \cdots\cdots$ for $x \in (-1,1)$

② $\frac{3x^2}{(1-x^3)^2} = 3x^2 + 6x^5 + 9x^8 + \cdots$ for $x \in (-1,1)$

Why?

We can use the previous two tools to create new power series from known ones.

e.g. we know $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x \in \underbrace{(-1,1)}_{\text{interval of convergence}}$

i.e. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots$ for $x \in (-1,1)$

Then, we get

① $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots\cdots$ for $x \in (-1,1)$] why?

② $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots\cdots$ for $x \in (-1,1)$

③ $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\cdots$ for $x \in [-1,1]$] why?

① If we know $f(x)=\sum_{n=0}^{\infty} a_n x^n$ then what are a_n?

Theorem Let $f(x)=a_0+a_1x+a_2x^2+\cdots\cdots$ be defined on an interval $(-R,R)$, then

$$a_n=\frac{f^{(n)}(0)}{n!}$$

[$f^{(k)}(x)$ denotes the k^{th} derivative of f]

Taylor series for f

② Does the power series $\boxed{\sum_{n=0}^{\infty} a_n x^n \text{ where } a_n=\frac{f^{(n)}(0)}{n!}}$ converge to $f(x)$ on some interval?

In other words, when does the Taylor series (or, Mclaurin series) of f actually equal f?

Are they always equal?

Consider $\sum_{n=0}^{\infty} a_n x^n$ where $a_n = \frac{f^{(n)}(0)}{n!}$ Taylor series of f

Let $S_N(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_N x^N$ seq. of partial sums of Taylor series of f

Ques Does $\lim_{N\to\infty} S_N(x) = f(x)$?

That is, $\boxed{E_N(x) = f(x) - S_N(x) \longrightarrow 0 \text{ as } N \to \infty \ ?}$ Error function

Can we give an alternate / useful description of the Error function $E_N(x)$?

Theorem [Lagrange Remainder Thm]

Let f be differentiable $N+1$ times on $(-R,R)$.
Define $a_n = \dfrac{f^{(n)}(0)}{n!}$ for $n=0,1,\dots,N$, and let

$$S_N(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_N x^N$$

Given $x \neq 0$ in $(-R,R)$, $\exists\, c \in (-|x|,|x|)$ such that the Error function $E_N(x) = f(x) - S_N(x)$ satisfies

$$\boxed{E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!}\, x^{N+1}}$$

As long as this makes sense, use it to check if $E_N(x) \to 0$ as $N \to \infty$

Proof Note that $f^{(n)}(0) = S_N^{(n)}(0)$ for $0 \le n \le N$

so, $E_N^{(n)}(0) = 0$ for all $n = 0, 1, 2, \ldots, N$

- Apply GMVT to the functions $E_N(x)$ and x^{N+1} in $[0, x]$

so, $\exists\ x_1 \in (0, x)$ s.T. $\boxed{\dfrac{E_N(x)}{x^{N+1}} = \dfrac{E_N'(x_1)}{(N+1)x_1^N}}$.

- Now, apply GMVT to $E_N'(x)$ and $(N+1)x^N$ on $[0, x_1]$

to get $x_2 \in (0, x_1)$ s.T. $\boxed{\dfrac{E_N'(x_1)}{(N+1)x_1^N} = \dfrac{E_N''(x_2)}{(N+1)N\, x_2^{N-1}}}$

- Continue . . . $\boxed{\dfrac{E_N(x)}{x^{N+1}} = \dfrac{E^{(N+1)}(x_{N+1})}{(N+1)!}}$ where $x_{N+1} \in (0, x_N) \subseteq \cdots \subseteq (0, x)$

Set $c = x_{N+1}$. we get $E_N(x) = \dfrac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$ $\left(\because E^{(N+1)}(x) = f^{(N+1)}(c)\right)$

<u>example</u> Taylor Series for $\sin(x)$

$a_0 = \sin(0) = 0$; $a_1 = \cos(0) = 1$; $a_2 = \frac{-\sin(0)}{2!} = 0$; $a_3 = \frac{-\cos(0)}{3!} = \frac{-1}{3!}$

we get $\quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$

How well does $S_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ approximate $\sin(x)$ on $[-2,2]$

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How well does $S_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ approximate $\sin(x)$ on $[-2,2]$

By Lagrange, $E_5(x) = \sin(x) - S_5(x) = -\frac{\sin(c)}{6!} x^6$ for some $c \in (-|x|, |x|)$

$$|E_5(x)| = \frac{|\sin(c)|}{6!} |x^6| \le \frac{1}{6!}(2^6) = \frac{2^6}{6!} \approx 0.089$$

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Does $S_N(x)$ converge uniformly to $\sin(x)$ on $[-2,2]$?

$$|E_N(x)| = \left|\frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}\right| \leq \frac{1}{(N+1)!}2^{N+1} \longrightarrow 0 \text{ uniformly on } [-2,2]$$

why?

example Taylor series for e^x is

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\cdots$$

But does it equal e^x?

Let $h(x)=e^x$ and $x_0\in[-R,R]$ for any $R>0$.

By Lagrange, $E_N(x_0)=\frac{h^{(N+1)}(c)}{(N+1)!}x_0^{N+1}$ for some $c\in(0,x_0)\subseteq[-R,R]$

Since $|h^{(N+1)}(c)|\leq e^c\leq e^R$, we get

$$|E_N(x_0)|\leq\left|\frac{e^R}{(N+1)!}x_0^{N+1}\right|\leq e^R\frac{R^{N+1}}{(N+1)!}\longrightarrow 0 \text{ as } N\to\infty$$

on $[-R,R]$

$$\therefore e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots \text{ on } \mathbb{R}$$

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on $[-R,R]$

$\therefore e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots$ on $\mathbb{R}$

Show this gives $e^{ix}=\cos x+i\sin x$ & $\boxed{e^{i\pi}+1=0}$

It is possible for a function (even if its infinitely differentiable) to ~~not~~ not equal its Taylor series, even if the Taylor series is convergent !!

Example $g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

Note $g(x) \neq 0 \quad \forall x \neq 0$

Show that $g^{(n)}(0) = 0 \quad \forall n$ [First try $g'(0)$ using L'Hospital's rule & then generalize]

This means the Taylor series of g has all coefficients equal to 0. Hence its convergent to 0.

But it does not equal $g(x)$, except at $x = 0$.

Math 400

Real Analysis

Video # 33

How to define an integral?

① Integrals find Antiderivatives.

- recall: term-by-term antidifferentiation of power series
- aim: FTOC $\int_a^b F'(x)\,dx = F(b) - F(a)$
  $G(x) = \int_a^x f(t)\,dt \Rightarrow G'(x) = f(x)$ ← Dem

Caution Darboux' Thm says every derivative satisfies IVP
consider $f(x)$ with a jump discontinuity e.g. $f(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 2 & \text{if } x \in [1,2] \end{cases}$
Then $f(x)$ cannot be equal to a derivative
i.e., $f(x) \neq G'(x)$, i.e., $\int_a^x f(t)\,dt$ does not exist

Integral purely as an antiderivative will limit what functions we can integrate.

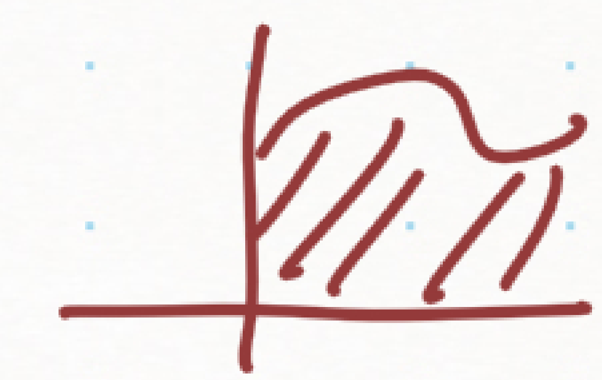
How to define an integral?

① Integrals find Antiderivatives.

- recall: term-by-term antidifferentiation of power series
- aim: FTOC $\int_a^b F'(x)\,dx = F(b) - F(a)$

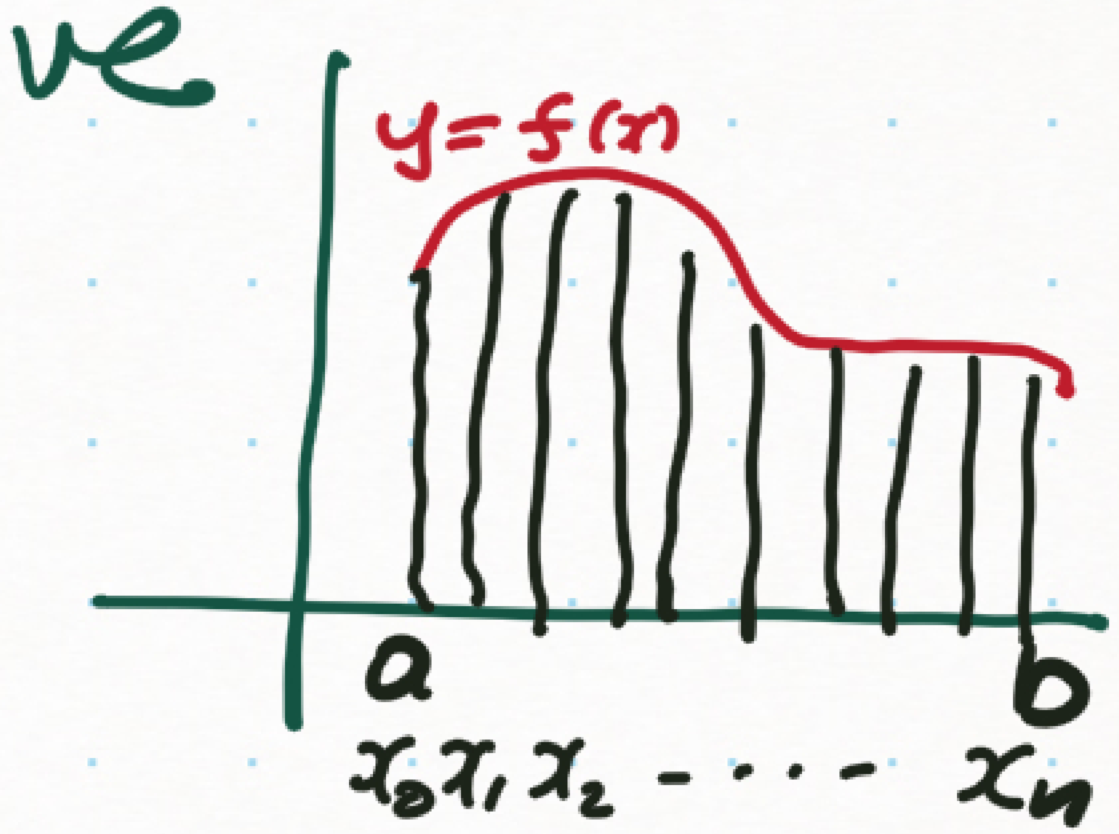
$$G(x) = \int_a^x f(t)\,dt \Rightarrow G'(x) = f(x)$$

② Integrals find Area under a curve



③ Be able to integrate as many functions as possible

- Riemann/Darboux Integral; Riemann-Stieltjes Integral; Lebesgue Integral; Daniell integral; Haar integral; Itô Integral; Stratonovich Integral; Young integral; .....



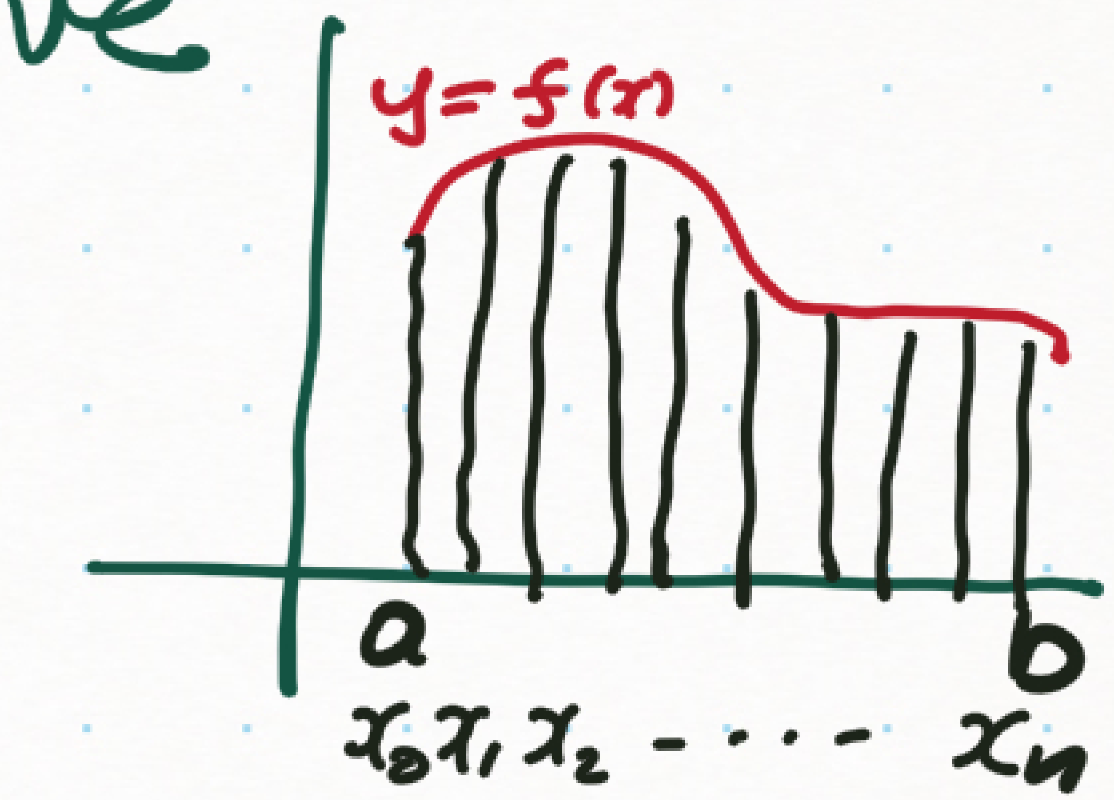
Riemann Integral as area under a curve

Let $f : [a,b] \to \mathbb{R}$ be a bounded function.

Partition P of $[a,b]$ is a finite set $P = \{x_0, x_1, \dots, x_n\}$ such that $x_0 = a$, $x_n = b$, and $x_0 < x_1 < x_2 < \dots < x_n$

P partitions $[a,b]$ into n subintervals: $[x_0, x_1], [x_2, x_3], \dots, [x_{n-1}, x_n]$

Riemann Integral as area under a curve



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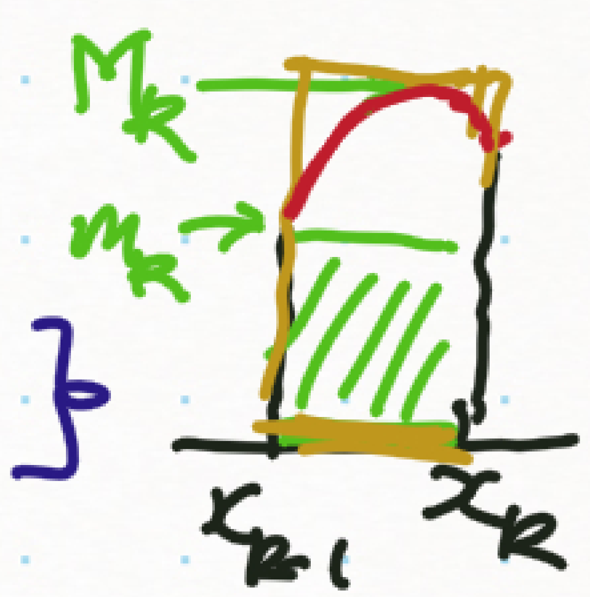
Partition P of $[a,b]$ is a finite set $P = \{x_0, x_1, \ldots, x_n\}$ such that

$x_0 = a$, $x_n = b$, and $x_0 < x_1 < x_2 < \cdots < x_n$

P partitions $[a,b]$ into n subintervals: $[x_0, x_1], [x_2, x_3], \ldots, [x_{n-1}, x_n]$

For each $[x_{k-1}, x_k]$ of P, let $m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}$

$M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$



Lower Sum of f w.r.t. P is $L(f,P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$

Upper Sum of f w.r.t. P is $U(f,P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$

$L(f,P) \le U(f,P)$

Given a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a,b]$.
A partition Q of $[a,b]$ is called a refinement of P if $P \subseteq Q$

Lemma [Refinements Refine]

If $P \subseteq Q$ then $L(f,P) \leq L(f,Q)$ and $U(f,Q) \leq U(f,P)$

Given a partition $P=\{x_0, x_1, \ldots, x_n\}$ of $[a,b]$.
A partition Q of $[a,b]$ is called a refinement of P if $Q \subseteq P$.

Lemma [Refinements Refine]

If $P \subseteq Q$ then $L(f,P) \leq L(f,Q)$ and $U(f,Q) \leq U(f,P)$

Proof: Since $P \subseteq Q$ and both P & Q are finite sets, we have $|Q \setminus P|$ is finite. So we can repeat the following process $k = |Q \setminus P|$ times (or simply do induction on k):

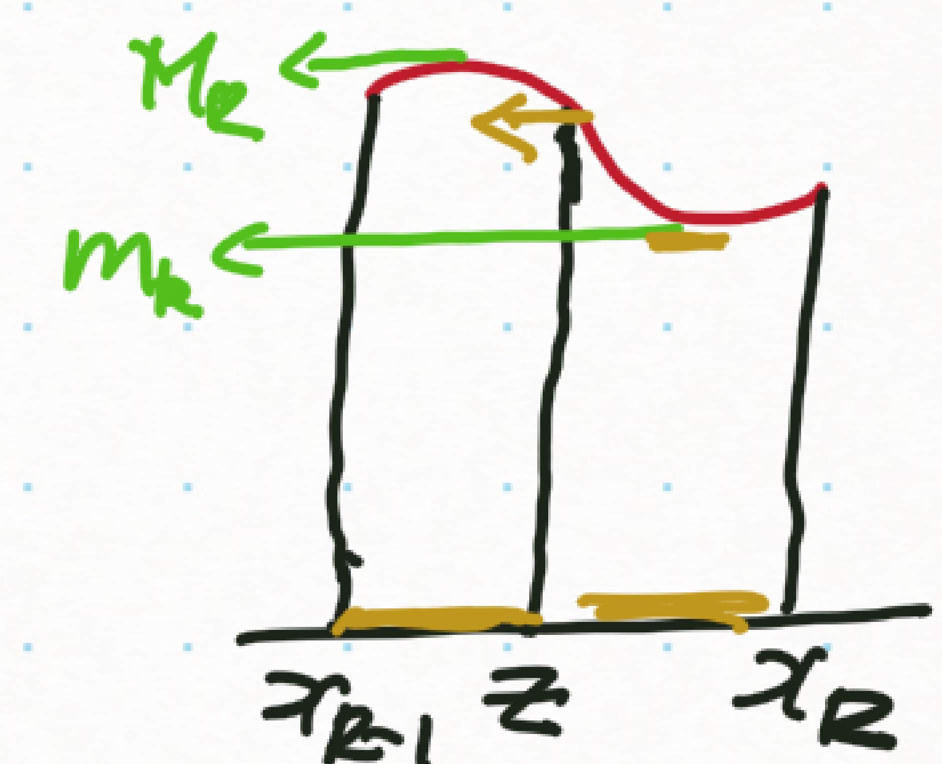
Let $z \in Q \setminus P$ then $z \in [x_{k-1}, x_k]$ for one specific k, $1 \leq k \leq n$.

In Lower sum: $m_k(x_k - x_{k-1}) = m_k(x_k - z) + m_k(z - x_{k-1})$

$$\leq m_k'(x_k - z) + m_k''(z - x_{k-1})$$

where $m_k' = \inf\{f(x) : x \in [z, x_k]\}$
$m_k'' = \inf\{f(x) : x \in [x_{k-1}, z]\}$

each is $\geq m_k$
$m_k' \geq m_k$ & $m_k'' \geq m_k$



Lemma [Lower Sums $\leq$ Upper Sums] $f:[a,b]\to\mathbb{R}$

① If P is a partition of $[a,b]$ then $L(f,P)\leq U(f,P)$

② If P_1 and P_2 are any two partitions of $[a,b]$ then $L(f,P_1)\leq U(f,P_2)$

Proof ① By definition.

② Let $Q=P_1\cup P_2$, the common refinement of P_1 & P_2.

So $P_1\subseteq Q$ and $P_2\subseteq Q$.

Applying previous lemma,

$$L(f,P_1)\leq L(f,Q)\overset{①}{\leq} U(f,Q)\leq U(f,P_2)$$

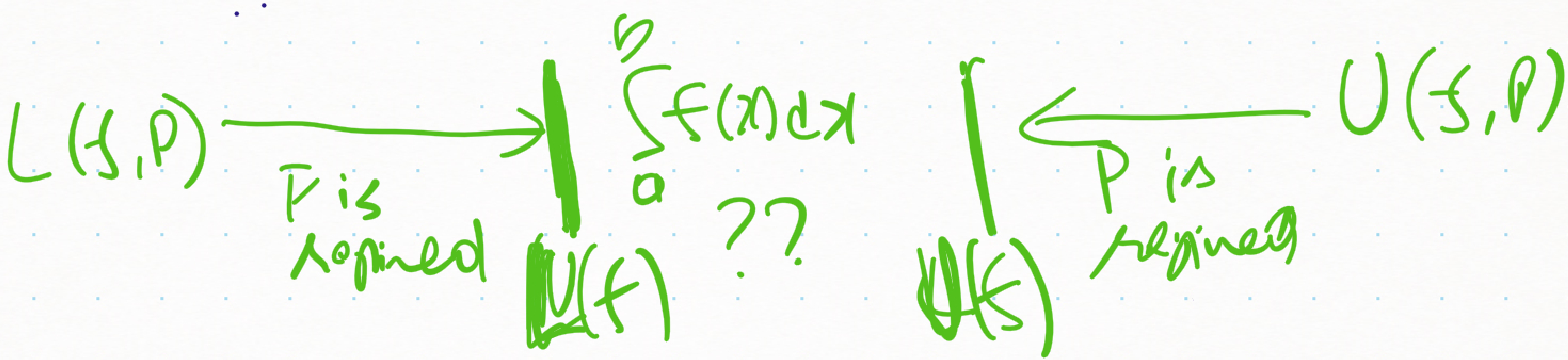
$$f:[a,b]\to\mathbb{R}$$

As partitions get more refined, our lower estimate increases and our upper estimate decreases, i.e. $L(f,P)$ & $U(f,P)$ get closer.

<u>**Defn**</u> Let $\mathcal{P}$ be the collection of all possible partitions of $[a,b]$.

<u>Upper Integral of f</u>, $U(f) = \inf\{U(f,P) : P\in\mathcal{P}\}$

<u>Lower Integral of f</u>, $L(f) = \sup\{L(f,P) : P\in\mathcal{P}\}$



$f : [a,b] \to \mathbb{R}$

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<u>Defn</u> Let $f : [a,b] \to \mathbb{R}$ be a bounded function.
We say f is <u>Riemann integrable</u> if $U(f) = L(f)$.
In this case we denote: $\int_a^b f = U(f) = L(f)$.

Proposition Let $f:[a,b]\to\mathbb{R}$ be a bounded function, say $m \le f(x) \le M$ $\forall x \in [a,b]$. Then

$$m(b-a) \le \underline{L(f) \le U(f)} \le M(b-a)$$

Proof $L(f) = \sup\{L(f,P) : P \in \mathcal{P}\} \le \inf\{U(f,P) : P \in \mathcal{P}\} = U(f)$

why? we know $L(f,P) \le U(f,P')$ for all $P, P' \in \mathcal{P}$

Show: If $x \in A$ and $y \in B \Rightarrow x \le y$ then $\sup A \le \inf B$

TRY IT!

Proposition Let $f:[a,b]\to\mathbb{R}$ be a bounded function, say $m\le f(x)\le M$ $\forall x\in[a,b]$. Then

$$m(b-a)\le L(f)\le U(f)\le M(b-a)$$

Proof $L(f)=\sup\{L(f,P):P\in\mathcal{P}\}\le\inf\{U(f,P):P\in\mathcal{P}\}=U(f)$

(why? we know $L(f,P)\le U(f,P')$ for all $P,P'\in\mathcal{P}$

Show: If $x\in A$ and $y\in B\Rightarrow x\le y$ then $\sup A\le\inf B$

Let $P_0=\{a,b\}$ be the 2-point partition of $[a,b]$ ($n=1$ & $x_0=a$, $x_1=b$)

$$\begin{aligned}L(f)&=\sup\{L(f,P):P\in\mathcal{P}\}\\&\ge L(f,P_0)\\&=\sum_{i=1}^{n}m_i(x_i-x_{i-1})\\&=m_1(b-a)\ge m(b-a)\end{aligned}$$

$$\begin{aligned}U(f)&=\inf\{U(f,P):P\in\mathcal{P}\}\\&\le U(f,P_0)\\&=\sum_{1}^{n}M_i(x_i-x_{i-1})\\&=M_1(b-a)\le M(b-a)\end{aligned}$$

Examples

① $f:[a,b]\to\mathbb{R}$ given by $f(x)=c$ $\forall x\in[a,b]$

we have $m=c\leq f(x)\leq c=M$.

By previous Prop., $c(b-a)\leq L(f)\leq U(f)\leq c(b-a)$

That is $L(f)=U(f)$

So $\int_a^b f(x)\,dx$ exists & equals $c(b-a)$

② Let $f: [0,1] \to \mathbb{R}$ be the Dirichlet Function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Recall: its discontinuous everywhere

② Let $f : [0,1] \to \mathbb{R}$ be the Dirichlet Function :

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$] Recall: its discontinuous everywhere

<u>Claim</u> f is <u>not</u> integrable

<u>Proof</u> Let P be an arbitrary partition of $[0,1]$

Each subinterval $[x_{k-1}, x_k]$ of P must contain both rationals and irrationals (by density of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$)

So, $m_k = 0$ and $M_k = 1 \quad \forall k$

$$\therefore L(f,P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1}) = \sum_{k=1}^{n} 0\,(x_k - x_{k-1}) = 0$$

$$U(f,P) = \sum M_k (x_k - x_{k-1}) = \sum 1\,(x_k - x_{k-1}) = \cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + \cancel{\cdots} + x_n - \cancel{x_{n-1}}$$
$$= x_n - x_0 = 1 - 0 = 1$$

So, $L(f) = \sup\{L(f,P) : P \in \mathcal{P}\} = \sup\{0\} = 0$

$U(f) = \inf\{U(f,P) : P \in \mathcal{P}\} = \inf\{1\} = 1$ not equal.

Math 400

Real Analysis

Video #34

Are all continuous functions integrable?
What about discontinuous functions?

$\sup\{L(f,P)\}$ $\inf\{U(f,P)\}$

We know $L(f) \leq U(f)$ always

But, f integrable $\iff$ $L(f) = U(f)$, i.e. we want elements of $U(f)$ to be arbitrarily close to elements of $L(f)$

Recall $a = b \iff \forall \epsilon > 0 \ |a-b| < \epsilon$

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Theorem [Integrability Criterion]

Let $f : [a,b] \to \mathbb{R}$ be bounded.

f is integrable $\iff$ $\forall \epsilon > 0 \ \exists$ partition P_ϵ of $[a,b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$

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Corollary Let $f : [a,b] \to \mathbb{R}$ be bounded

f is integrable $\iff$ $\exists$ sequence of partitions P_n of $[a,b]$ s.t. $\lim\limits_{n\to\infty} \left[U(f, P_n) - L(f, P_n) \right] = 0$

[See Exercise 7.2.3]

$$\overset{\sup\{L(f,P)\}}{L(f)} \leq \overset{\inf\{U(f,P)\}}{U(f)}$$

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Proof $\boxed{\Leftarrow}$

Given $\epsilon > 0$, suppose such a partition P_ϵ exists.

Then $U(f) - L(f) \leq U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$

Then $U(f) = L(f)$ (by char. of $a=b$ above).

$\boxed{\Rightarrow}$ f is integrable, so $L(f) = U(f)$.

since $U(f)$ is the inf, i.e. greatest lower bound of all $U(f, P)$

we know, Given $\epsilon > 0$, $\exists$ a partition P_1 s.t.

$$U(f, P_1) < \underline{U(f)} + \epsilon/2$$

Similarly, for $L(f)$ as the sup, we get a partition P_2 s.t.

$$L(f, P_2) < L(f) - \epsilon/2$$

$\boxed{\Rightarrow}$ f is integrable, so $L(f) = U(f)$.

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we know, Given $\epsilon > 0$, $\exists$ a partition P_1 s.t.

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Similarly, for $L(f)$ as the sup, we get a partition P_2 s.t.

$$L(f, P_2) < L(f) - \epsilon/2 \quad \text{---} \; (**)$$

Let $P_\epsilon = P_1 \cup P_2$, the common refinement.

$$U(f, P_\epsilon) - L(f, P_\epsilon) \leq U(f, P_1) - L(f, P_2) \quad \text{[Since "Refinements Refine"]}$$

$$< \left(U(f) + \epsilon/2\right) - \left(L(f) - \epsilon/2\right) \; \text{---} \text{by } (*) \;\&\; (**)$$

$$= \int_a^b f(x)dx + \epsilon/2 - \int_a^b f(x)dx + \epsilon/2 = \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Theorem [Continuous $\Rightarrow$ Integrable]

If $f:[a,b]\to\mathbb{R}$ is continuous, then f is integrable.

Proof f continuous on compact set $\Rightarrow$ f is bounded and unif. cont..

Given $\epsilon>0$, since f is unif. cont., $\exists\,\delta>0$ s.t.

$$|f(x)-f(y)|<\frac{\epsilon}{b-a} \quad \text{for all } |x-y|<\delta \text{ and all } x,y\in[a,b]$$

Let P_ϵ be a partition of $[a,b]$ s.t. $\Delta x_k = x_k - x_{k-1} < \delta \ \forall k$.

In each subinterval of P_ϵ, $[x_{k-1},x_k]$, EVT tells us that

$$\left.\begin{aligned} M_k &= \sup_{[x_{k-1},x_k]} f = f(z_k) \text{ for some } z_k\in[x_{k-1},x_k] \\ m_k &= \inf_{[x_{k-1},x_k]} f = f(y_k) \text{ for some } y_k\in[x_{k-1},x_k] \end{aligned}\right\} \Rightarrow |z_k-y_k|<\delta$$

$$\Downarrow$$

$$\boxed{M_k-m_k = f(z_k)-f(y_k)} < \epsilon/(b-a)$$

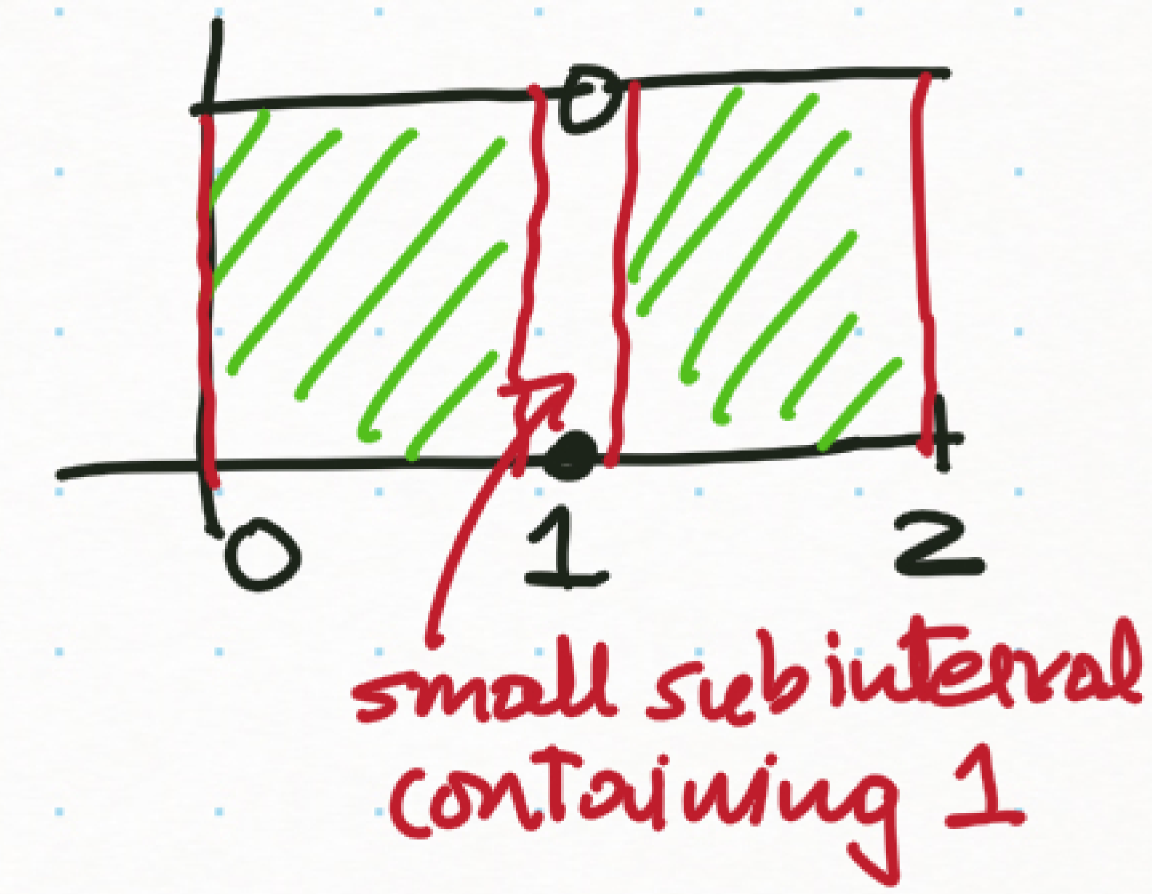
Finally, $U(f,P_\epsilon)-L(f,P_\epsilon)=\sum(M_k-m_k)\Delta x_k < \frac{\epsilon}{b-a}\sum\Delta x_k$

$$=\frac{\epsilon}{b-a}(b-a)=\epsilon$$

What about discontinuous functions?

Example

① $f(x) : [0,2] \to \mathbb{R}$ as $f(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$



Let $P_\epsilon = \{0, 1-\frac{\epsilon}{4}, 1+\epsilon/4, 2\}$ $n=3$

In $[0, 1-\epsilon/4]$, $m_1 = 1$ and $M_2 = 1$.

In $[1-\epsilon/4, 1+\epsilon/4]$, $m_2 = 0$ and $M_2 = 1$.

In $[1+\epsilon/4, 2]$, $m_3 = 1$ and $M_3 = 1$.

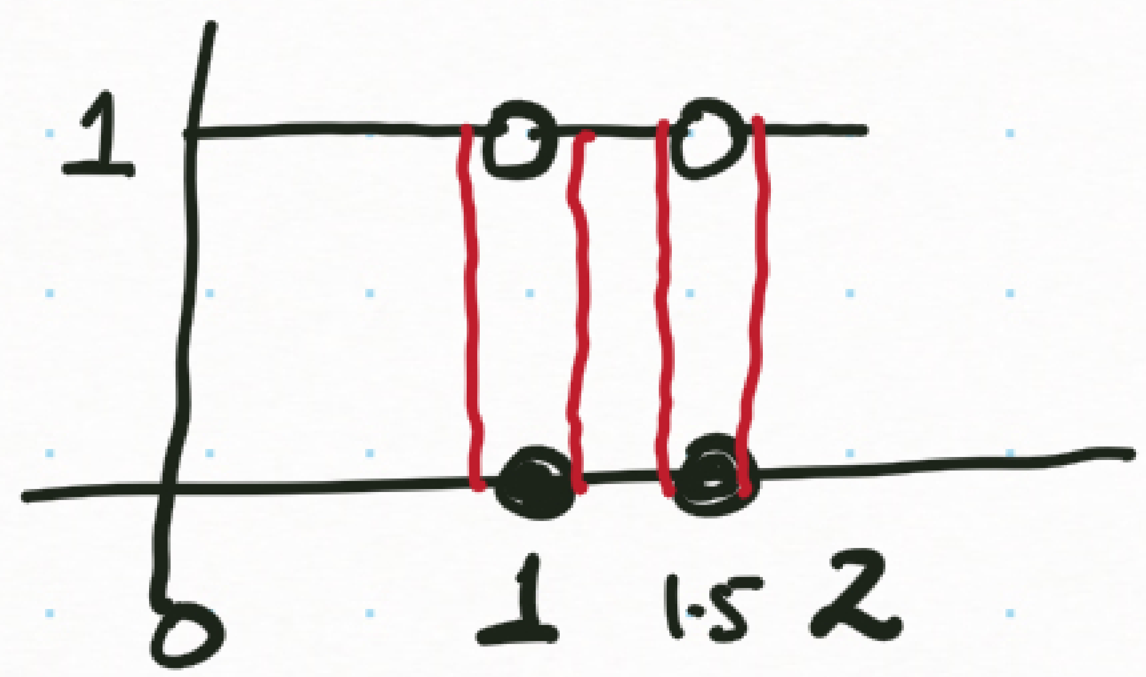
$$U(f,P_\epsilon) = \sum_{k=1}^{3} M_k (x_k - x_{k-1}) = 1\left(1-\frac{\epsilon}{4}-0\right) + 1\left(1+\frac{\epsilon}{4} - \left(1-\frac{\epsilon}{4}\right)\right) + 1\left(2-\left(1+\frac{\epsilon}{4}\right)\right)$$
$$= 1 - \frac{\epsilon}{4} + 1 + \frac{\epsilon}{4} - 1 + \frac{\epsilon}{4} + 2 - 1 - \frac{\epsilon}{4} = 2$$

$$L(f,P_\epsilon) = \sum m_k (x_k - x_{k-1}) = 1\left(1-\frac{\epsilon}{4}-0\right) + 0\left(1+\frac{\epsilon}{4} - \left(1-\frac{\epsilon}{4}\right)\right) + 1\left(2-\left(1-\frac{\epsilon}{4}\right)\right)$$
$$= 1 - \epsilon/4 + 2 - 1 - \epsilon/4 = 2 - \epsilon/2$$

$\therefore U(f,P_\epsilon) - L(f,P_\epsilon) = 2 - (2 - \epsilon/2) = \epsilon/2 < \epsilon$ $\therefore \int_0^2 f(x)\,dx$ exists & equals 2.

What if f has 2 discontinuities?

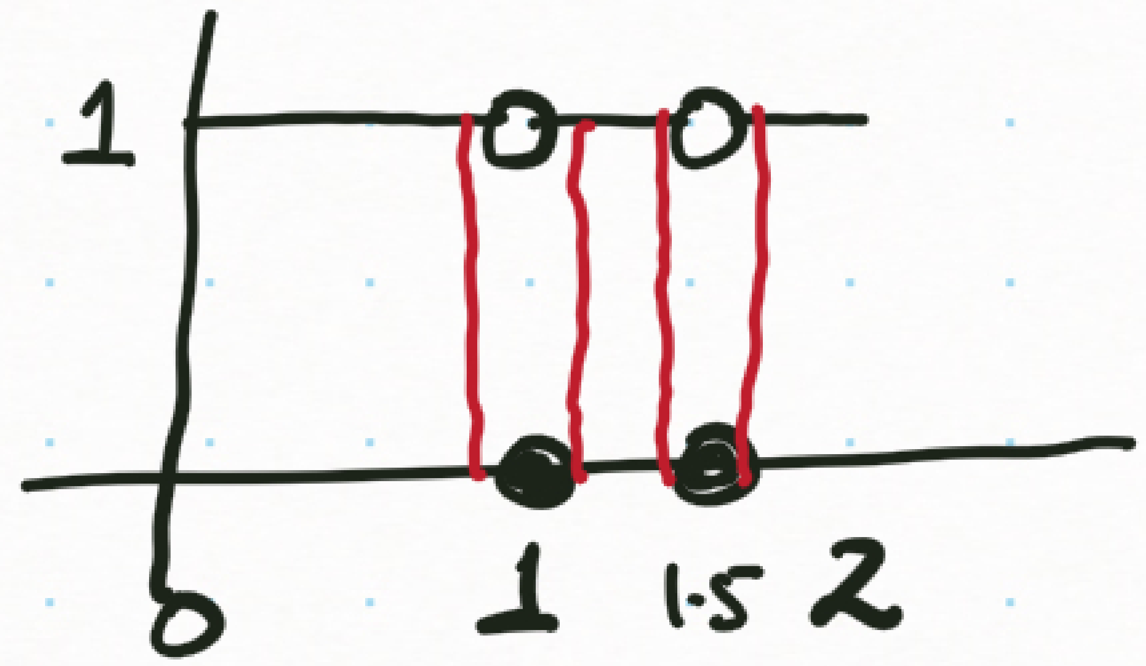
(2) Let $f:[0,2]\to\mathbb{R}$ as $f(x)=\begin{cases}1 & \text{if } x\neq 1 \text{ or } 1.5\\ 0 & \text{if } x=1 \text{ or } 1.5\end{cases}$



Define P_ϵ as?

What if f has 2 discontinuities?

(2) Let $f : [0,2] \to \mathbb{R}$ as $f(x) = \begin{cases} 1 & \text{if } x \neq 1 \text{ or } 1.5 \\ 0 & \text{if } x = 1 \text{ or } 1.5 \end{cases}$



Define P_ϵ as? $P_\epsilon = \{0, 1-\frac{\epsilon}{8}, 1+\frac{\epsilon}{8}, 1.5-\frac{\epsilon}{8}, 1.5+\frac{\epsilon}{8}, 2\}$ $n=5$

Check (same as example (1)),

$U(f, P_\epsilon) = 2$

and $L(f, P_\epsilon) = 2 - \frac{\epsilon}{2}$

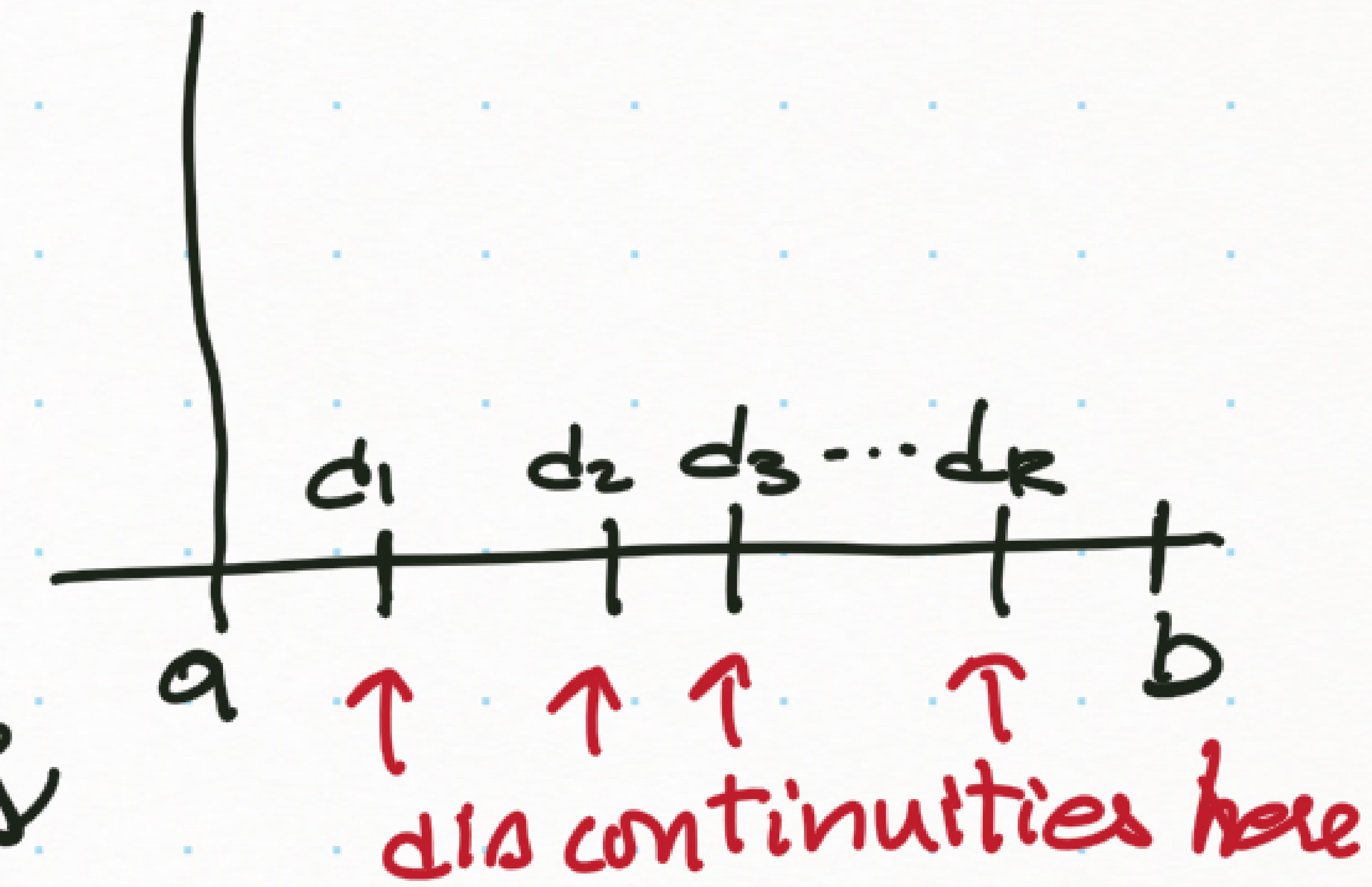
$\Big\}$ so $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$

What if f has k discontinuities?

We could repeat the previous arguments by defining a partition P_ϵ that "isolates" each discontinuity.

Is there an easier / cleaner way?

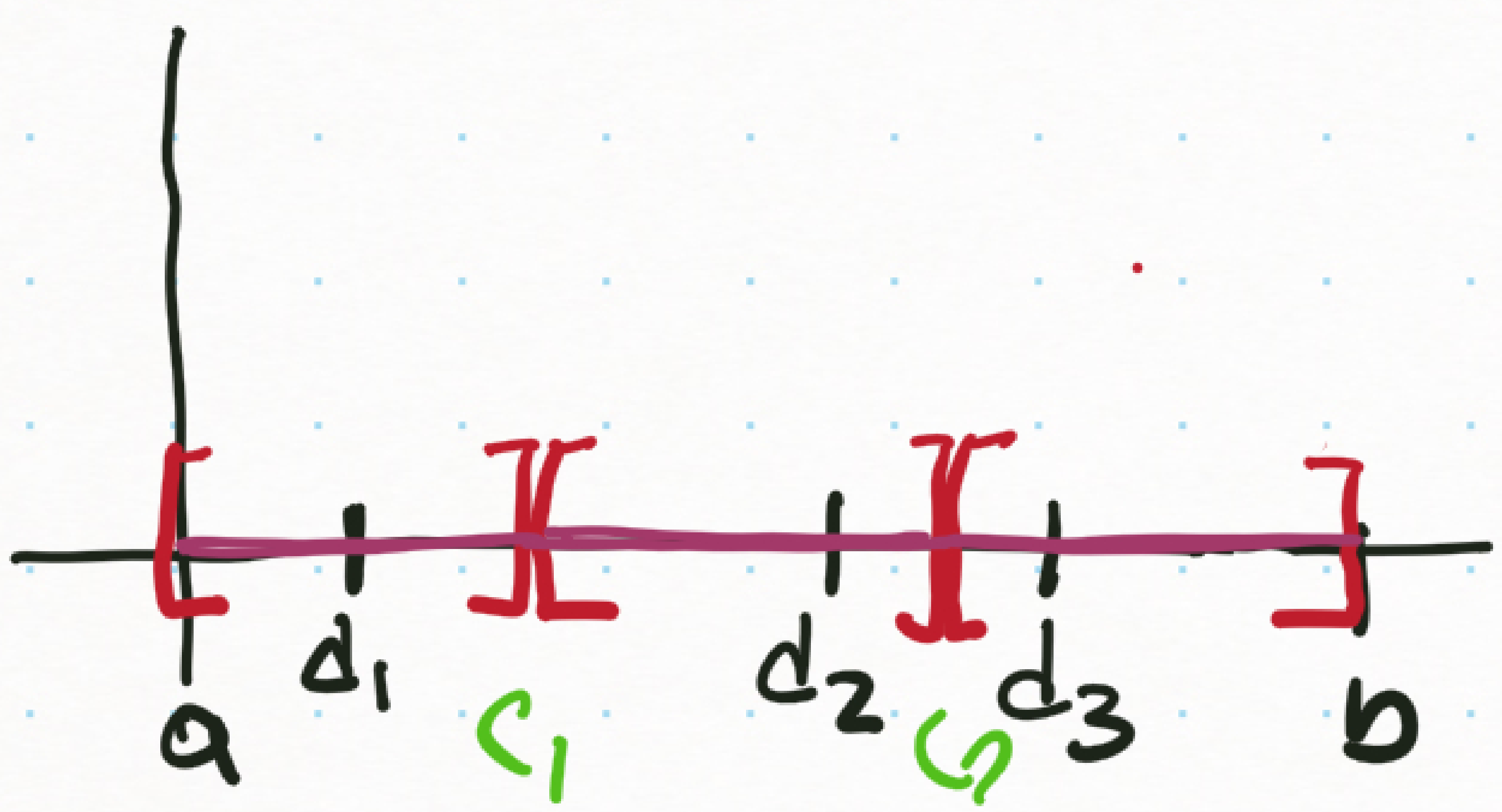
What if f has k discontinuities?



Using the characterization of integrability we can prove:

Theorem Let $f:[a,b]\to\mathbb{R}$ and $a<c<b$.

f is integrable on $[a,b] \iff f$ is integrable on both $[a,c]$ and $[c,b]$.



If f has discontinuities on points d_1, d_2, d_3 in $[a,b]$, then check f is integrable on 3 intervals each containing exactly one d_i.