Math 400

Real Analysis

Video #35

We already saw (two) arguments for showing a function with finitely many discontinuities is integrable.

We also saw that a function with infinitely many discontinuities need not be integrable. [Dirichlet function]

Is it possible for some function with infinitely many discontinuities to be integrable?

Example $f:[0,2]\to\mathbb{R}$ with $f(x)=\begin{cases}1 & \text{if } x\neq\frac{1}{n} \text{ for all } n\in\mathbb{N}\\ 0 & \text{if } x=\frac{1}{n} \text{ for some } n\in\mathbb{N}\end{cases}$

f has countably many discontinuities, at each point of the $1/n$, $n\in\mathbb{N}$.

Example $f:[0,2] \to \mathbb{R}$ with $f(x)=\begin{cases} 1 & \text{if } x \neq \frac{1}{n} \text{ for all } n \in \mathbb{N} \\ 0 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \end{cases}$

f has countably many discontinuities, at each point $\frac{1}{n}, n \in \mathbb{N}$

Given $\epsilon > 0$, we want to define a partition P_ϵ of $[0,2]$ such that one very small subinterval of P_ϵ "takes care" of most of the discontinuities, leaving only finitely many for the remaining finitely many subintervals of P_ϵ.

Example

$f : [0,2] \to \mathbb{R}$ with $f(x) = \begin{cases} 1 & \text{if } x \neq \frac{1}{n} \text{ for all } n \in \mathbb{N} \\ 0 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N} \end{cases}$

f has countably many discontinuities, at each point $\frac{1}{n}, n \in \mathbb{N}$

Given $\epsilon > 0$, by Archimedean Principle $\exists$ (smallest) N s.t. $\frac{1}{N+1} < \epsilon/4$.

Define $P_\epsilon = \left\{0, \frac{\epsilon}{4}, \frac{1}{N} - \frac{\epsilon}{8N}, \frac{1}{N} + \frac{\epsilon}{8N}, \frac{1}{N-1} - \frac{\epsilon}{8N}, \frac{1}{N-1} + \frac{\epsilon}{8N}, \ldots, \frac{1}{1} - \frac{\epsilon}{8N}, \frac{1}{1} + \frac{\epsilon}{8N}, 2\right\}$

Example $f:[0,2]\to\mathbb{R}$ with $f(x)=\begin{cases}1 & \text{if } x\neq \frac{1}{n} \text{ for all } n\in\mathbb{N}\\ 0 & \text{if } x=\frac{1}{n} \text{ for some } n\in\mathbb{N}\end{cases}$

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infinitely many discontinuities $\frac{1}{N+1}, \frac{1}{N+2}, \frac{1}{N+3}, \dots\dots$

$\frac{1}{N}$ discont

$\frac{1}{N-1}$ discont

. . . .

1 discont

. . .

No Discontinuities

Example $f:[0,2]\to\mathbb{R}$ with $f(x)=\begin{cases}1 & \text{if } x\neq \frac{1}{n} \text{ for all } n\in\mathbb{N}\\ 0 & \text{if } x=\frac{1}{n} \text{ for some } n\in\mathbb{N}\end{cases}$

f has countably many discontinuities, at each point $\frac{1}{n}$, $n\in\mathbb{N}$

Given $\epsilon>0$, by Archimedean Principle $\exists$ (smallest) N s.t. $\frac{1}{N+1}<\epsilon/4$.

Define
$$P_\epsilon=\{\underbrace{0,\tfrac{\epsilon}{4}}_{\substack{\text{infinitely many discontinuities}\\ \frac{1}{N+1},\frac{1}{N+2},\frac{1}{N+3},\dots}},\ \underbrace{\tfrac{1}{N}-\tfrac{\epsilon}{8N},\ \tfrac{1}{N}+\tfrac{\epsilon}{8N}}_{\frac{1}{N}\text{ discont}},\ \underbrace{\tfrac{1}{N-1}-\tfrac{\epsilon}{8N},\ \tfrac{1}{N-1}+\tfrac{\epsilon}{8N}}_{\frac{1}{N-1}\text{ discont}},\ \dots\dots,\ \underbrace{\tfrac{1}{1}-\tfrac{\epsilon}{8N},\ \tfrac{1}{1}+\tfrac{\epsilon}{8N}}_{1\text{ discont}},2\}$$

. . . .

. . .

NO Discontinuities

In these subintervals, $m_k=M_k=1$, so $U(f,P_\epsilon)=L(f,P_\epsilon)$ in these subintervals

Example $f:[0,2]\to\mathbb{R}$ with $f(x)=\begin{cases}1 & \text{if } x\neq \frac{1}{n} \text{ for all } n\in\mathbb{N}\\ 0 & \text{if } x=\frac{1}{n} \text{ for some } n\in\mathbb{N}\end{cases}$

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Define $P_\epsilon=\{0,\underbrace{\frac{\epsilon}{4}}_{\text{infinitely many discontinuities } \frac{1}{N+1},\frac{1}{N+2},\dots},\frac{1}{N}-\frac{\epsilon}{8N},\underbrace{\frac{1}{N}+\frac{\epsilon}{8N}}_{\frac{1}{N}\text{ discont}},\frac{1}{N-1}-\frac{\epsilon}{8N},\underbrace{\frac{1}{N-1}+\frac{\epsilon}{8N}}_{\frac{1}{N-1}\text{ discont}},\dots,\frac{1}{1}-\frac{\epsilon}{8N},\underbrace{\frac{1}{1}+\frac{\epsilon}{8N}}_{1\text{ discont}},2\}$

infinitely many discontinuities $\frac{1}{N+1},\frac{1}{N+2},\dots$

$$U(f,P_\epsilon)-L(f,P_\epsilon)=\sum_{k=1}^{N+1}(M_k-m_k)(x_k-x_{k-1})$$
$$=(1-0)(\epsilon/4-0)+\sum_{k=2}^{N+1}(1-0)\left(\frac{\epsilon}{4N}\right)$$
$$=\frac{\epsilon}{4}+N(1-0)\left(\frac{\epsilon}{4N}\right)=\frac{\epsilon}{4}+\frac{\epsilon}{4}=\frac{\epsilon}{2}<\epsilon$$

$\boxed{\text{Therefore, } f \text{ is integrable}}$

<u>Another Example</u> Thomae's function has countably many discontinuities, one at each rational number, but it is integrable over any finite interval.
[Exercise 7.3.2]

Can some function with uncountably many discontinuities be integrable?

Recall $\mathcal{C}$ = Cantor Set.

Define $f(x) = \begin{cases} 1 & \text{if } x \in \mathcal{C} \\ 0 & \text{if } x \notin \mathcal{C} \end{cases}$

f has uncountably many discontinuities
But it can be shown f is integrable.

Lebesgue characterized which discontinuous functions are integrable. [See Section 7.6 ← Optional]

Lebesgue's Integrability Criterion

Assume $f:[a,b]\to\mathbb{R}$ is a bounded function, and D be the set of all points where f is discontinuous. Then, f is integrable $\Longleftrightarrow$ D has measure zero.

Defn set D has measure zero if $\forall \epsilon>0$, $\exists$ countable collection of intervals $I_1, I_2, I_3, \cdots$ such that

$$\underline{D \subseteq \bigcup_{k=1}^{\infty} I_k} \quad \text{and} \quad \underline{\sum_{k=1}^{\infty} L(I_k) < \epsilon},$$

where $L(I_k)$ is the length of interval I_k

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Theorem [Basic Properties of Riemann Integral]

Assume functions f and g are integrable on $[a,b]$.

① Function $f+g$ is integrable on $[a,b]$ with $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

② For any $k \in \mathbb{R}$, kf is integrable with $\int_a^b (kf) = k \int_a^b f$

③ If $f(x) \leq g(x)$ on $[a,b]$ then $\int_a^b f \leq \int_a^b g$

④ Function $|f|$ is integrable with $\left| \int_a^b f \right| \leq \int_a^b |f|$

[We already know: $m \leq f(x) \leq M \ \forall x \in [a,b] \Rightarrow m(b-a) \leq \int_a^b f \leq M(b-a)$

In particular, if $f(x) \geq 0 \ \forall x \in [a,b]$ then $\int_a^b f \geq 0$

Proofs

Proofs of 1 & 2 use the corollary to the Integrability criterion (see lecture notes or Exercise 7.2.3 from HW):

f is integrable on $[a,b] \iff \exists$ seq. of partitions (P_n) of $[a,b]$ s.t. $\lim_{n\to\infty}[U(f,P_n)-L(f,P_n)]=0$

and then $\int_a^b f = \lim U(f,P_n) = \lim L(f,P_n)$

Proofs

Proofs of ① & ② use the corollary to the Integrability criterion (see lecture notes or Exercise 7.2.3 from HW):

f is integrable on $[a,b] \iff \exists$ seq. of partitions (P_n) of $[a,b]$ s.t. $\lim\limits_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0$

and then $\int_a^b f = \lim U(f,P_n) = \lim L(f,P_n)$

What if $k<0$?

To prove ②: For $k \geq 0$, Observe each m_i for f will transform to km_i for kf

each M_i for f will change to kM_i for kf

So, $U(kf,P) = k\,U(f,P)$ and $L(kf,P) = k\,L(f,P)$ $\forall$ partitions P.

Since f is integrable, $\exists\ (P_n)$ s.t. $\int_a^b f = \lim U(f,P_n) = \lim L(f,P_n)$

By Algebra of limits, $\lim\limits_{n\to\infty} U(kf,P_n) = \lim k\,U(f,P_n) = k \lim U(f,P_n) = k\int_a^b f$

$\lim\limits_{n\to\infty} L(kf,P_n) = \lim k\,L(f,P_n) = k \lim L(f,P_n) = k\int_a^b f$

Proof of ③ [$f \le g \Rightarrow \int f \le \int g$]

$f \le g \Rightarrow g - f \ge 0$. $\quad$ f integrable $\overset{\text{by ②}}{\Rightarrow}$ $(-1)f$ integrable, ie., $-f$ int.

$\therefore g - f = g + (-f)$ is also integrable by ①.

$\therefore \int_a^b (g-f) \ge 0$, by earlier property since $g-f \ge 0$ & $g-f$ int.

Now, again by ① & ②,

$\int_a^b (g-f) \ge 0 \Rightarrow \int_a^b g - \int_a^b f \ge 0 \Rightarrow \int_a^b g \ge \int_a^b f$.

Integral Triangle Inequality

Proof of (4) [$|f|$ integrable & $\left|\int_a^b f\right| \le \int_a^b |f|$]

[Exercise 7.4.1] f integrable $\Rightarrow$ f bdd. $\Rightarrow$ $|f|$ bdd.

f integrable $\Rightarrow$ $\exists$ partition $P_\epsilon = \{x_0, \dots, x_n\}$ s.t. $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$

Show (i) $U(|f|, P_\epsilon) - L(|f|, P_\epsilon) < \epsilon$ (showing $|f|$ is integrable)

by first showing: (ii) $m_k \le \overline{m_k}$ and $M_k \le \overline{M_k}$

where $\overline{m_k} = \inf\{|f(x)| : x \in [x_{k-1}, x_k]\}$

$\overline{M_k} = \sup\{|f(x)| : x \in [x_{k-1}, x_k]\}$

and consequently, (iii) $M_k - m_k \ge \overline{M_k} - \overline{m_k}$

And finally use (3) to show (iv) $\left|\int_a^b f\right| \le \int_a^b |f|$ on $-|f| \le f \le |f|$

Definition For any function f, $\int_a^a f = 0$

For any integrable $f : [a,b] \to \mathbb{R}$, $\int_a^b f = -\int_b^a f$

Using above conventions, we can now write

$$\int_a^b f = \int_a^c f + \int_c^b f$$

for any three points a, b, c chosen in any order from an interval I over which f is integrable.

Thm [Integrable Limit Theorem]

Assume $f_n \to f$ uniformly on $[a,b]$, and each f_n is integrable on $[a,b]$

Then, f is integrable and $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$

Proof Given $\epsilon > 0$, $\exists N$ s.t. $|f_N(x) - f(x)| \leq \frac{\epsilon}{3(b-a)}$ $\forall x \in [a,b]$ — (1)

f_N integrable $\Rightarrow \exists$ partition P_ϵ s.t. $U(f_N, P_\epsilon) - L(f_N, P_\epsilon) < \frac{\epsilon}{3}$ — (2)

Let $M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}$ & $N_k = \sup\{f_N(x) : x \in [x_{k-1}, x_k]\}$ over the subintervals of P_ϵ.

By (1), $|M_k - N_k| \leq \epsilon/3(b-a)$. which gives us $\rightarrow$

$|U(f, P_\epsilon) - U(f_N, P_\epsilon)| = \left|\sum (M_k - N_k)(x_k - x_{k-1})\right| \leq \sum \frac{\epsilon}{3(b-a)} \Delta x_k = \epsilon/3$ — (3)

Similarly, $|L(f, P_\epsilon) - L(f_N, P_\epsilon)| \leq \epsilon/3$ — (4)

Plugging in ①, ②, ③, ④ into

$$U(f,P_\epsilon)-L(f,P_\epsilon)=\boxed{\begin{aligned}&|U(f,P_\epsilon)-U(f_N,P_\epsilon)\\&+U(f_N,P_\epsilon)-L(f_N,P_\epsilon)\\&+L(f_N,P_\epsilon)-L(f,P_\epsilon)|\end{aligned}}\le \begin{aligned}&|U(f,P_\epsilon)-U(f_N,P_\epsilon)|\\&+|U(f_N,P_\epsilon)-L(f_N,P_\epsilon)|\\&+|L(f_N,P_\epsilon)-L(f,P_\epsilon)|\end{aligned}$$

$$\le \epsilon/3+\epsilon/3+\epsilon/3=\epsilon$$

$\therefore$ f is integrable.

property #4

By properties of integral, $\left|\int_a^b f_n-\int_a^b f\right|=\left|\int_a^b (f_n-f)\right|\le\int_a^b|f_n-f|$

Given $\epsilon>0$, $\exists N$ s.t. $|f_n(x)-f(x)|<\epsilon/(b-a)$ $\forall n\ge N$ & $x\in[a,b]$

Thus, for $n\ge N$, $\left|\int_a^b f_n-\int_a^b f\right|\le\int_a^b|f_n-f|\le\int_a^b\frac{\epsilon}{b-a}=\epsilon$

$$\therefore \lim_{n\to\infty}\int_a^b f_n=\int_a^b f.$$

Integral Mean Value Theorem

If f is a continuous function on $[a,b]$, then $\exists\, c \in [a,b]$ s.t. $f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx$

Proof

[HW] What is $\frac{1}{b-a}\int_a^b f(x)\,dx$? And in which interval does it lie?

Apply IVT to f and

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Fundamental Theorem of Calculus

(i) If $f:[a,b] \longrightarrow \mathbb{R}$ is integrable, and $F:[a,b] \to \mathbb{R}$ satisfies
$F'(x) = f(x) \ \forall x \in [a,b]$

then $$\int_a^b f = F(b) - F(a)$$

(ii) Let $g:[a,b] \longrightarrow \mathbb{R}$ be integrable, and
for $x \in [a,b]$ define $G(x) = \int_a^x g$

Then G is continuous on $[a,b]$.

If g is continuous at some point $c \in [a,b]$
then G is differentiable at c with $G'(c) = g(c)$.

Applications of FTOC

[Integration by Parts] If f and g be differentiable with f' and g' continuous on $[a,b]$, then fg' and $f'g$ are integrable and

$$\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g$$

Recall from Calculus: $\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$

Proof is simply integrate over the product rule for differentiation & apply FTOC (i).

Applications of FTOC

[Integration by Parts] If f and g be differentiable with f' and g' continuous on $[a,b]$, then fg' and $f'g$ are integrable and

$$\boxed{\int_a^b fg' = f(b)g(b) - f(a)g(a) - \int_a^b f'g}$$

Proof fg' & $f'g$ are continuous and hence integrable.

By Product Rule, $(fg)' = f'g + fg'$

Integrating, $\int_a^b (fg)' = \int_a^b f'g + \int_a^b fg'$

By FTOC, $\int_a^b (fg)' = (fg)(b) - (fg)(a) = f(b)g(b) - f(a)g(a)$

$\therefore\ f(b)g(b) - f(a)g(a) = \int_a^b f'g + \int_a^b fg'$.

Example

Evaluate $\int_0^{\pi} x \cos x \, dx$

$f(x)$ $\qquad$ $g'(x)$ $\qquad$ & Both f & g are continuously differentiable.

i.e., $g(x) = \sin x$

Applying Int. by parts,

$$\int_0^{\pi} x \cos x \, dx = f(\pi)\, g(\pi) - f(0)\, g(0) - \int_0^{\pi} f'(x)\, g(x)\, dx$$

$$= \pi \sin \pi - 0 \sin 0 - \int_0^{\pi} 1 \sin x \, dx$$

$$= -\int_0^{\pi} \sin x \, dx$$

$$= -2$$

[Substitution Rule] If g is differentiable and g' is continuous on $[a,b]$ and f is continuous on $g([a,b])$

Then,

$$\boxed{\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(t)\, dt}$$

Proof is to apply chain rule to $F(g(x))$ where $F(x) = \int_{g(a)}^{x} f(t)\, dt$

and simplify with FTOC (ii)

<u>Example</u> Simplify $\boxed{\int_a^b h(x)\,dx + \int_{h(a)}^{h(b)} h^{-1}(u)\,du}$

where $h:(a,b)\to\mathbb{R}$ is a 1-to-1 differentiable function.

Note $h^{-1}: h((a,b))\to\mathbb{R}$ is well defined on range of h, $h((a,b))$, and is differentiable.

Apply <u>Substitution Rule</u> with $f=h^{-1}$ and $g=h$,

$$\int_a^b h^{-1}(h(x))\,h'(x)\,dx = \int_{h(a)}^{h(b)} h^{-1}(t)\,dt$$

i.e., $\int_{h(a)}^{h(b)} h^{-1}(t)\,dt = \int_a^b x\,h'(x)\,dx$

$= b\,h(b) - a\,h(a) - \int_a^b h(x)\,dx$, by <u>integration by parts</u>

i.e., $\boxed{\int_a^b h(x)\,dx + \int_{h(a)}^{h(b)} h^{-1}(u)\,du = b\,h(b) - a\,h(a)}$ (!!)

Proof of FTOC (i)

f integrable & $F' = f$ on $[a,b]$
Then, $\int_a^b f = F(b) - F(a)$

Let $P = \{a = x_0 < x_1 < \dots < x_n = b\}$ be any partition of $[a,b]$.

Since F is differentiable, by MVT applied to F on each $[x_k, x_{k+1}]$

we get $\exists\, c_k \in (x_{k-1}, x_k)$ s.t. $F'(c_k) = \dfrac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}}$

i.e., $F(x_k) - F(x_{k-1}) = F'(c_k)(x_k - x_{k-1})$

$= f(c_k)(x_k - x_{k-1})$ — ⊛ ($\because F' = f$)

Now, $L(f,P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1}) \le \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1}) \le \sum_{k=1}^{n} M_k (x_k - x_{k-1})$

i.e., $L(f,P) \le \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) \le U(f,P)$ (using ⊛)

i.e., $L(f,P) \le F(b) - F(a) \le U(f,P)$ (Telescoping sum)

i.e., $\int_a^b f \le F(b) - F(a) \le \int_a^b f$ (since f is integrable, $U(f) = L(f) = \int_a^b f$)

i.e., $\int_a^b f = F(b) - F(a)$

Proof of FTOC (ii)

g integrable & $G(x)=\int_a^x g(t)\,dt$
Then G is continuous on $[a,b]$.
Moreover, if g continuous then G diff. & $G'=g$

Assume g integrable.

If $g(x)=0\ \forall x$ then $G(x)=0\ \forall x$ & we are done.

So, we may assume $M=\sup\{|g(x)| : x\in[a,b]\}$ is >0.

For any $x_0\in[a,b]$, we show G is continuous at x_0.

Given $\epsilon>0$, let $\delta=\epsilon/M$

why? why? why?

$$|G(x)-G(x_0)| = \left|\int_a^x g - \int_a^{x_0} g\right| = \left|\int_{x_0}^x g\right| \le \int_{x_0}^x |g| \le M|x-x_0|$$

$$< M\delta \quad \text{when } |x-x_0|<\delta$$

$$= M\frac{\epsilon}{M}$$

$$=\epsilon$$

$\therefore$ G is cont. at x_0.

Assume g is also continuous.

For any $c \in [a,b]$, we have to show $\lim_{x\to c} \frac{G(x)-G(c)}{x-c} = g(c)$.

equivalently, for any $(x_n) \subseteq [a,b] \setminus \{c\}$, $x_n \to c \Rightarrow \lim_{n\to\infty} \frac{G(x_n)-G(c)}{x_n - c} = g(c)$

Note,
$$G(x_n) - G(c) = \int_a^{x_n} g - \int_a^c g = \int_c^{x_n} g,$$

Using Integral MVT [f continuous on $[a,b] \Rightarrow \exists\, c \in [a,b] : f(c) = \frac{1}{b-a}\int_a^b f$]

we get, $\int_c^{x_n} g = g(c_n)(x_n - c)$, i.e, $G(x_n) - G(c) = g(c_n)(x_n - c)$

where c_n lies between x_n & c

i.e, $\frac{G(x_n)-G(c)}{x_n - c} = g(c_n)$

Now, $G'(c) = \lim_{n\to\infty} \frac{G(x_n)-G(c)}{x_n - c} = \lim_{n\to\infty} g(c_n)$

$= g\left(\lim_{n\to\infty} c_n\right)$ ($\because$ g is continuous)

$= g(c)$ ($\because$ c_n lies between x_n & c, apply Squeeze)