Math 400

Real Analysis

Part #31

Please review "Series of numbers".

<u>Defn</u> Let (f_n) be a sequence of functions on $A \subseteq \mathbb{R}$.
Let $S_k(x) = f_1(x) + f_2(x) + \dots + f_k(x)$ be seq. of partial sums.

① The series $\sum_{n=1}^{\infty} f_n(x)$ converges pointwise to $f : A \to \mathbb{R}$
if $S_k(x)$ converges pointwise to $f(x)$.

② The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A to $f : A \to \mathbb{R}$
if $S_k(x)$ converges uniformly on A to $f(x)$.

Everything we want to understand about $\sum_{n=1}^{\infty} f_n(x) = f(x)$ reduces to understand how the sequence of partial sums $(S_k(x))$ behaves under the mode of convergence (pointwise or uniform).

Theorem Let $f_n : A \to \mathbb{R}$ be continuous $\forall n$.
Assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to a function f.
Then, f is continuous on A.

Everything we want to understand about $\sum_{n=1}^{\infty} f_n(x) = f(x)$ reduces to understand how the sequence of partial sums $(S_k(x))$ behaves under the mode of convergence (pointwise or uniform).

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Assume $\sum_{n=1}^{\infty} f_n$ converges uniformly on A to a function f.
Then, f is continuous on A.

<u>Proof</u> f_n continuous functions $\Rightarrow$ $S_k = \sum_{i=1}^{k} f_i$ is continuous (Why?)
By Continuous limit Thm., f is continuous since $S_k(x) \to f(x)$ uniformly on A.

Theorem Let $f_n : I \to \mathbb{R}$ be differentiable on interval I $\forall n$.
Assume $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly to $g(x)$ on I.
If $\exists\, x_0 \in I$ s.t. $\sum_{n=1}^{\infty} f_n(x_0)$ converges, then

$\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to $f(x)$ on I

and $f(x)$ is differentiable with $f'(x) = g(x)$ on I.

i.e., $f(x) = \sum_{n=1}^{\infty} f_n(x)$ and $f'(x) = \sum_{n=1}^{\infty} f_n'(x)$

Proof follows directly from Differentiable Limit Theorem (Stronger) applied to $S_k(x)$.

Theorem [Cauchy Criterion for Unif. Convergence of Series]

Series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $A \subseteq \mathbb{R}$

$\Longleftrightarrow$ $\forall \epsilon > 0$, $\exists N$ s.t. $|f_{m+1}(x) + \dots + f_n(x)| < \epsilon$ $\forall n > m \geq N$ and $x \in A$.

Theorem [Cauchy Criterion for Unif. Convergence of Series]

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Cor [Weierstrass M-Test]

For each n, let $f_n : A \to \mathbb{R}$ and $M_n \in \mathbb{R}^+$ s.t. $|f_n(x)| \leq M_n$ $\forall x \in A$.

If $\sum_{n=1}^{\infty} M_n$ converges then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on A.

Proof [Fill in details]

$\sum_{n=1}^{\infty} M_n$ converges $\Rightarrow$ partial sum of (M_n) are Cauchy $\overset{?}{\Rightarrow}$ $\sum f_n(x)$ is Cauchy $\Rightarrow$ $\sum f_n(x)$ converges unif.

<u>Example</u> Let $f_n(x) = \dfrac{1}{x^4 + 3x^2 n + n^2 + 7}$ on $\mathbb{R}$

What is behavior of $\sum_{n=1}^{\infty} f_n(x)$?

Example Let $f_n(x) = \dfrac{1}{x^4 + 3x^2 n + n^2 + 7}$ on $\mathbb{R}$

What is behavior of $\sum_{n=1}^{\infty} f_n(x)$?

$$|f_n(x)| = \frac{1}{x^4+3x^2n+n^2+7} \le \frac{1}{n^2} \quad \forall x$$

(since every term in denominator is nonnegative)

Let $M_n = \dfrac{1}{n^2}$ Then $\sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$ converges by series p-test ($p=2$)

$\therefore$ by Weierstrass M-Test, $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $\mathbb{R}$.

Power series is a function of the form $\sum_{n=0}^{\infty} a_n x^n$

For which values of x does a power series converge?

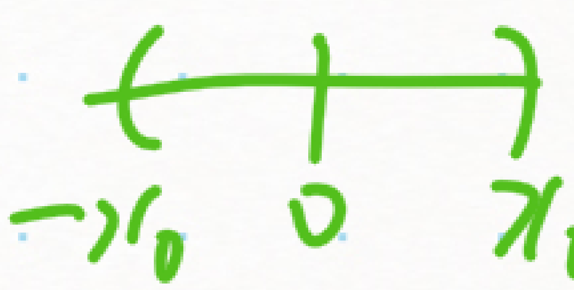
e.g. $\sum_{k=0}^{\infty} x^k$ is a power series with each $a_k = 1$.

It is a geometric series with common ratio $r = x$, that converges precisely when $r \in (-1, 1)$

$\therefore \sum_{k=0}^{\infty} x^k$ converges to $\frac{1}{1-x}$ when $x \in (-1, 1)$.

Theorem If $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for each x s.t. $|x| < |x_0|$.

$x \in (-|x_0|, |x_0|)$



Proof: Suppose $\sum_{n=0}^{\infty} a_n x_0^n$ converges, then the sequence of terms $(a_n x_0^n)$ converges to 0

so, $(a_n x_0^n)$ is bounded.

That is, $\exists M > 0$ s.t. $|a_n x_0^n| \leq M \ \forall n$

If $x \in \mathbb{R}$ satisfies $|x| < |x_0|$, Then

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M \left|\frac{x}{x_0}\right|^n \ \forall n$$

Note $\sum_{n=0}^{\infty} M \left|\frac{x}{x_0}\right|^n$ is a geometric series with $r = \left|\frac{x}{x_0}\right| < 1$ & converges.

By series comparison test, $\sum |a_n x^n|$ converges.

Theorem If $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$, then $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for each x s.t. $|x| < |x_0|$.

$\hookrightarrow$ $x \in (-|x_0|, |x_0|)$, an interval of values where $\sum a_n x^n$ converges

So, convergence of $\sum a_n x^n$ must occur on an interval

i.e., $\underset{\substack{\| \\ [0,0] \\ R=0}}{\{0\}}$, $\underset{\substack{\| \\ (-\infty,\infty) \\ R=\infty}}{\mathbb{R}}$, or $(-R,R)$ or $[-R,R)$ or $(-R,R]$ or $[-R,R]$.

The value R above is called Radius of Convergence of a power series.

e.g. $\sum x^n$ has $R=1$ (since its convergent on $(-1,1)$)

Theorem [Uniform Convergence of Power Series]

If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely at $x_0 \in \mathbb{R}$ then $\sum a_n x^n$ converges uniformly on $[-c, c]$ where $c = |x_0|$.

Proof Apply Weierstrass M-test

$\sum a_n x^n$ converges absolutely at x_0 means $\sum_{n=0}^{\infty} |a_n x_0^n|$ converges

Let $M_n = |a_n x_0^n|$ then $\sum M_n$ converges.

If $x \in [-c, c]$ then $|a_n x^n| \leq |a_n c^n| = |a_n x_0^n| = M_n \ \forall n$

So by Weierstrass M-test, $\sum a_n x^n$ converges uniformly on $[-c, c]$.

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Real Analysis

Part #32

Recall, we know

Thm If power series $\sum_{n=0}^{\infty} a_n x^n$ converges at some point $x_0 \in \mathbb{R}$ then it converges absolutely for every $x \in (-|x_0|, |x_0|)$ ← Interval of convergence

Thm If power series $\sum a_n x^n$ converges absolutely at a point x_0, then it converges uniformly on $[-|x_0|, |x_0|]$

Since each $a_n x^n$ is cont. & $\sum a_n x^n$ converges uniformly we get $\sum a_n x^n$ is also continuous.

Are power series differentiable? Is term-by-term differentiation allowed?

Yes, but proofs will be tedious unless we develop some more theory.

<u>Theorem</u> Suppose $f(x)=\sum_{n=0}^{\infty} a_n x^n$ converges on an interval $I \subseteq \mathbb{R}$. Then,

f is continuous on I and differentiable on any $(-R,R) \subseteq I$.

The derivative is given by $f'(x)=\sum_{n=1}^{\infty} n a_n x^{n-1}$.

Moreover, f is infinitely differentiable on $(-R,R)$ and the successive derivatives are obtained by term-by-term differentiation of the previous series.

What about "Integrability"?

↳ Antiderivative

<u>Theorem</u> Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges on $(-R, R)$

Then $F(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ is defined on $(-R, R)$

and $F'(x) = f(x)$ (That is, we can do term-by-term antidifferentiation

"Integration"

We can use the previous two tools to create new power series from known ones.

e.g. we know $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x \in \underbrace{(-1,1)}_{\text{interval of convergence}}$

i.e. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots\cdots$ for $x \in (-1,1)$

Then, we get

① $\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + \cdots\cdots$ for $x \in (-1,1)$

② $\frac{3x^2}{(1-x^3)^2} = 3x^2 + 6x^5 + 9x^8 + \cdots$ for $x \in (-1,1)$

Why?

We can use the previous two tools to create new power series from known ones.

e.g. we know $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $x \in \underbrace{(-1,1)}_{\text{interval of convergence}}$

i.e. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots\ldots$ for $x \in (-1,1)$

Then, we get

① $\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots\ldots$ for $x \in (-1,1)$ Why?

② $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - \ldots\ldots$ for $x \in (-1,1)$

③ $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots$ for $x \in (-1,1]$

Why?

① If we know $f(x) = \sum_{n=0}^{\infty} a_n x^n$ then what are a_n?

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots\cdots$$

$$f(0) = ?$$

$$f'(0) = ?$$

$$\vdots$$

① If we know $f(x)=\sum_{n=0}^{\infty} a_n x^n$ then what are a_n?

<u>Theorem</u> Let $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots\cdots$ be defined on an interval $(-R,R)$, then

$$a_n = \frac{f^{(n)}(0)}{n!}$$

[$f^{(k)}(x)$ denotes the k^{th} derivative of f]

Taylor series for f

② Does the power series $\boxed{\sum_{n=0}^{\infty} a_n x^n \text{ where } a_n = \frac{f^{(n)}(0)}{n!}}$ converge to $f(x)$ on some interval?

In other words, when does the Taylor series (or, Mclaurin series) of f actually equal f?

Are they always equal?

Consider $\sum_{n=0}^{\infty} a_n x^n$ where $a_n = \frac{f^{(n)}(0)}{n!}$ Taylor series of f

Let $S_N(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_N x$ seq. of partial sums of Taylor series of f

Ques Does $\lim_{N \to \infty} S_N(x) = f(x)$?

That is, $\boxed{E_N(x) = f(x) - S_N(x) \longrightarrow 0 \text{ as } N \to \infty \text{ ?}}$ Error function

Can we give an alternate / useful description of the Error function $E_N(x)$?

Theorem [Lagrange Remainder Thm]

Let f be differentiable $N+1$ times on $(-R,R)$.
Define $a_n = \frac{f^{(n)}(0)}{n!}$ for $n=0,1,\dots,N$, and let

$$S_N(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_N x^N$$

Given $x \neq 0$ in $(-R,R)$, $\exists\, c \in (-|x|,|x|)$ such that the Error function $E_N(x) = f(x) - S_N(x)$ satisfies

$$\boxed{E_N(x) = \frac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}}$$

As long as this makes sense, use it to check if $E_N(x) \to 0$ as $N \to \infty$

Proof Note that $f^{(n)}(0) = S_N^{(n)}(0)$ for $0 \le n \le N$

so, $E_N^{(n)}(0) = 0$ for all $n = 0, 1, 2, \dots, N$

- Apply GMVT to the functions $E_N(x)$ and x^{N+1} in $[0, x]$

so, $\exists\, x_1 \in (0, x)$ s.T. $\dfrac{E_N(x)}{x^{N+1}} = \dfrac{E_N'(x_1)}{(N+1)x_1^N}$.

- Now, apply GMVT to $E_N'(x)$ and $(N+1)x^N$ on $[0, x_1]$

to get $x_2 \in (0, x_1)$ s.T. $\dfrac{E_N'(x_1)}{(N+1)x_1^N} = \dfrac{E_N''(x_2)}{(N+1)N\, x_2^{N-1}}$

- Continue ... $\dfrac{E_N(x)}{x^{N+1}} = \dfrac{E^{(N+1)}(x_{N+1})}{(N+1)!}$ where $x_{N+1} \in (0, x_N) \subseteq \dots \subseteq (0, x)$

Set $c = x_{N+1}$. we get $E_N(x) = \dfrac{f^{(N+1)}(c)}{(N+1)!} x^{N+1}$ $\left(\because E^{(N+1)}(x) = f^{(N+1)}(c)\right)$

example Taylor Series for $\sin(x)$

$a_0 = \sin(0) = 0$; $a_1 = \cos(0) = 1$; $a_2 = \frac{-\sin(0)}{2!} = 0$; $a_3 = \frac{-\cos(0)}{3!} = \frac{-1}{3!}$

we get $\quad x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$

How well does $S_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ approximate $\sin(x)$ on $[-2,2]$

<u>example</u> Taylor Series for $\sin(x)$

$a_0 = \sin(0) = 0$; $a_1 = \cos(0) = 1$; $a_2 = -\dfrac{\sin(0)}{2!} = 0$; $a_3 = \dfrac{-\cos(0)}{3!} = \dfrac{-1}{3!}$

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By Lagrange, $E_5(x) = \sin(x) - S_5(x) = -\dfrac{\sin(c)}{6!} x^6$ for some $c \in (-|x|, |x|)$

$$|E_5(x)| = \frac{|\sin(c)|}{6!} |x^6| \leq \frac{1}{6!}(2^6) = \frac{2^6}{6!} \approx 0.089$$

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$$|E_5(x)| = \frac{|\sin(c)|}{6!}|x^6| \leq \frac{1}{6!}(2^6) = \frac{2^6}{6!} \approx 0.089$$

Does $S_N(x)$ converge uniformly to $\sin(x)$ on $[-2,2]$?

$$|E_N(x)| = \left|\frac{f^{(N+1)}(c)}{(N+1)!}x^{N+1}\right| \leq \frac{1}{(N+1)!}2^{N+1} \longrightarrow 0 \text{ uniformly on } [-2,2]$$

why?

<u>example</u> Taylor series for e^x is

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots\cdots$$

But does it equal e^x?

Let $h(x)=e^x$ and $x_0 \in [-R,R]$ for any $R>0$.

By Lagrange, $E_N(x_0)=\frac{h^{(N+1)}(c)}{(N+1)!}x_0^{N+1}$ for some $c \in (0,x_0) \subseteq [-R,R]$

Since $|h^{(N+1)}(c)| \leq e^c \leq e^R$, we get

$$|E_N(x_0)| \leq \left|\frac{e^R}{(N+1)!}x_0^{N+1}\right| \leq e^R\frac{R^{N+1}}{(N+1)!} \longrightarrow 0 \text{ as } N\to\infty$$

on $[-R,R]$

$$\therefore e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots \text{ on } \mathbb{R}$$

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But does it equal e^x?

Let $h(x)=e^x$ and $x_0 \in [-R,R]$ for any $R>0$.

By Lagrange, $E_N(x_0)=\frac{h^{(N+1)}(c)}{(N+1)!}x_0^{N+1}$ for some $c\in(0,x_0)\subseteq[-R,R]$

Since $|h^{(N+1)}(c)| \le e^c \le e^R$, we get

$$|E_N(x_0)| \le \left|\frac{e^R}{(N+1)!}x_0^{N+1}\right| \le e^R \frac{R^{N+1}}{(N+1)!} \longrightarrow 0 \text{ as } N\to\infty$$

on $[-R,R]$

$\therefore e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\cdots$ on $\mathbb{R}$

Show this gives $e^{ix}=\cos x+i\sin x$ & $\boxed{e^{i\pi}+1=0}$

It is possible for a function (even if its infinitely differentiable) to not equal its Taylor series, even if the Taylor series is convergent !!

Example
$$g(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Note $g(x) \neq 0 \quad \forall x \neq 0$

Show that $g^{(n)}(0) = 0 \quad \forall n$ [First try $g'(0)$ using L'Hospital's rule & then generalize]

This means the Taylor series of g has all coefficients equal to 0. Hence its convergent to 0.

But it does not equal $g(x)$, except at $x = 0$.

Math 400

Real Analysis

Part # 33

How to define an integral?

1 Integrals find Antiderivatives.

- recall: term-by-term antidifferentiation of power series
- aim: FTOC $\int_a^b F'(x)\,dx = F(b) - F(a)$

$$G(x) = \int_a^x f(t)\,dt \Rightarrow G'(x) = f(x)$$

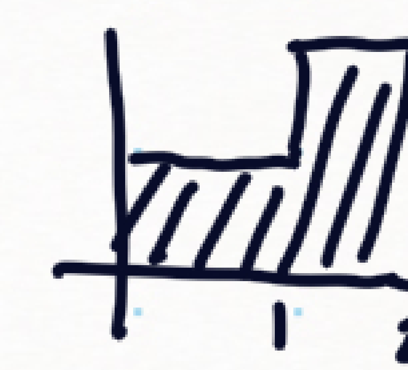
← Dem

Caution Darboux' Thm says every derivative satisfies IVP

consider $f(x)$ with a jump discontinuity e.g. $f(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ 2 & \text{if } x \in [1,2] \end{cases}$

Then $f(x)$ cannot be equal to a derivative

i.e., $f(x) \neq G'(x)$, i.e., $\int_a^x f(t)\,dt$ does not exist



Integral purely as an antiderivative will limit what functions we can integrate.

How to define an integral?

1. Integrals find Antiderivatives.

   - recall: term-by-term antidifferentiation of power series
   - aim: FTOC $\int_a^b F'(x)\,dx = F(b) - F(a)$

     $G(x) = \int_a^x f(t)\,dt \Rightarrow G'(x) = f(x)$

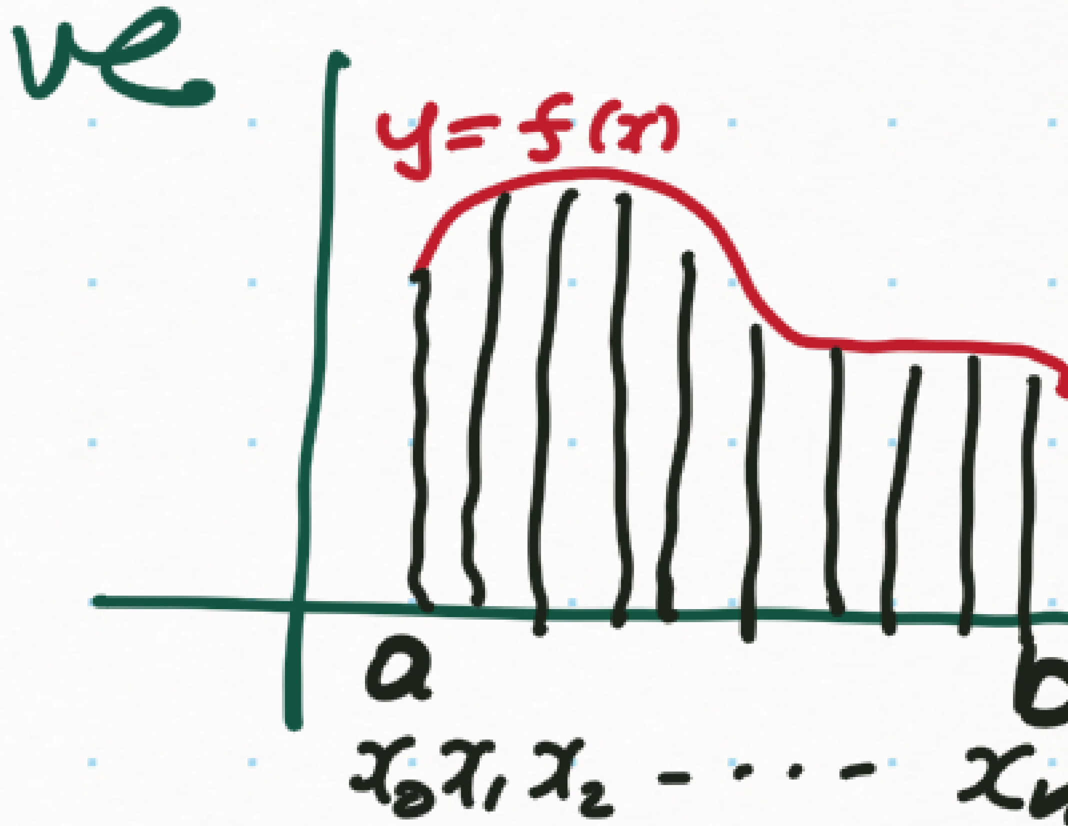
2. Integrals find Area under a curve

3. Be able to integrate as many functions as possible

   - Riemann/Darboux Integral; Riemann-Stieltjes Integral; Lebesgue Integral; Daniell integral; Haar integral; Ito Integral; Stratonovich Integral; Young integral; .....

Riemann Integral as area under a curve



Let $f:[a,b] \to \mathbb{R}$ be a bounded function.

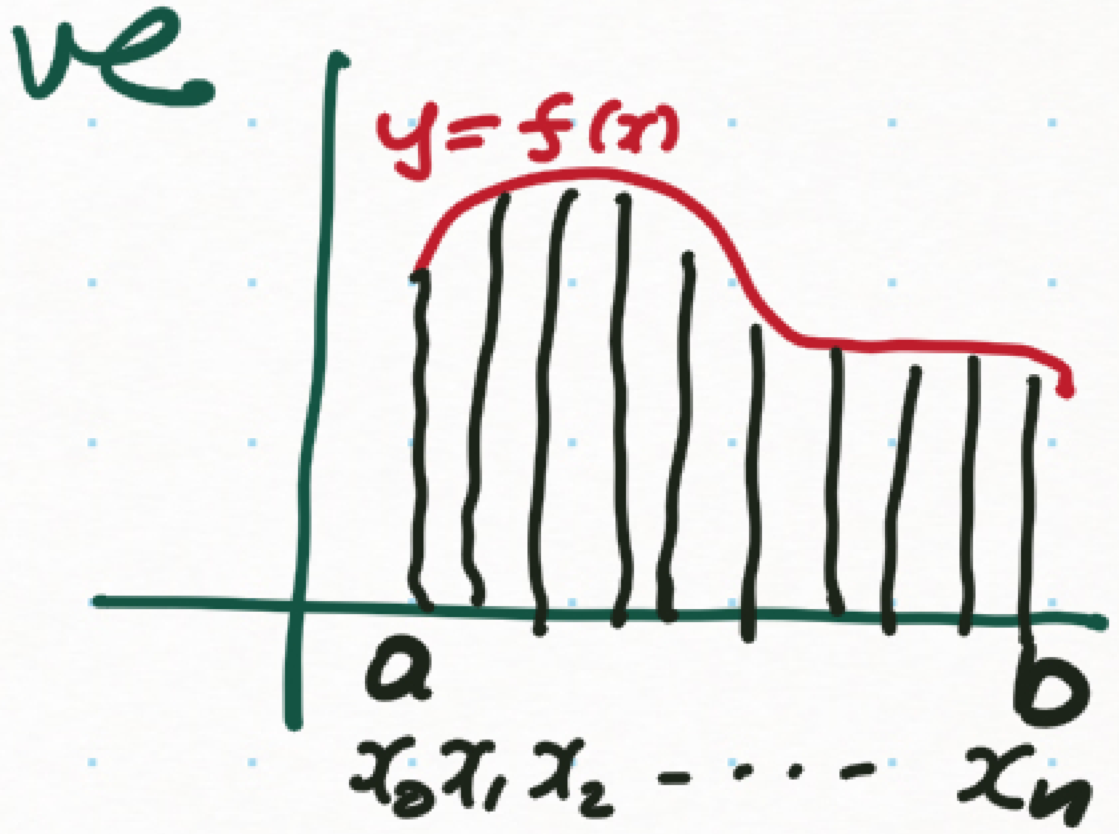
Partition P of $[a,b]$ is a finite set

$P = \{x_0, x_1, \ldots, x_n\}$ such that

$x_0 = a$, $x_n = b$, and $x_0 < x_1 < x_2 < \cdots < x_n$

P partitions $[a,b]$ into n subintervals: $[x_0, x_1], [x_2, x_3], \ldots, [x_{n-1}, x_n]$

Riemann Integral as area under a curve



Let $f:[a,b]\to\mathbb{R}$ be a bounded function.

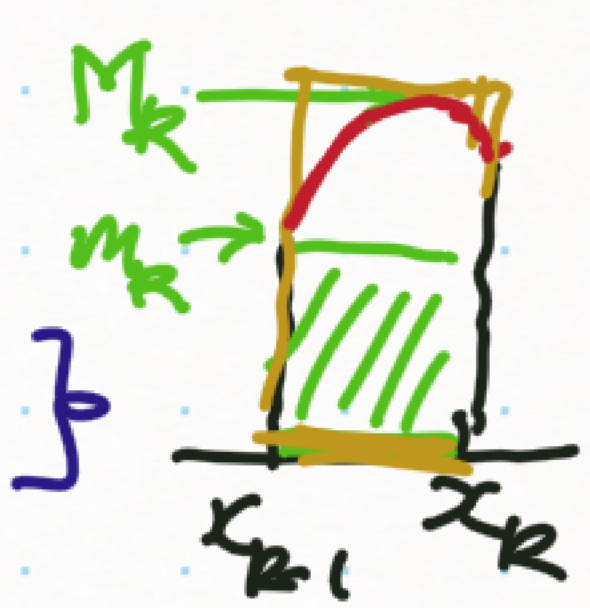
Partition P of $[a,b]$ is a finite set $P=\{x_0, x_1, \ldots, x_n\}$ such that

$x_0=a$, $x_n=b$, and $x_0 < x_1 < x_2 < \cdots < x_n$

P partitions $[a,b]$ into n subintervals: $[x_0,x_1],[x_2,x_3],\ldots,[x_{n-1},x_n]$

For each $[x_{k-1},x_k]$ of P, let $m_k = \inf\{f(x) : x\in[x_{k-1},x_k]\}$

$M_k = \sup\{f(x) : x\in[x_{k-1},x_k]\}$



Lower Sum of f w.r.t. P is $L(f,P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1})$

Upper Sum of f w.r.t. P is $U(f,P) = \sum_{k=1}^{n} M_k (x_k - x_{k-1})$

$L(f,P) \le U(f,P)$

Given a partition $P=\{x_0, x_1, \ldots, x_n\}$ of $[a,b]$.
A partition Q of $[a,b]$ is called a refinement of P if $P \subseteq Q$

Lemma [Refinements Refine]

If $P \subseteq Q$ then $L(f,P) \leq L(f,Q)$ and $U(f,Q) \leq U(f,P)$

Given a partition $P=\{x_0, x_1, \dots, x_n\}$ of $[a,b]$.
A partition Q of $[a,b]$ is called a refinement of P if $Q \subseteq P$.

Lemma [Refinements Refine]

If $P \subseteq Q$ then $L(f,P) \leq L(f,Q)$ and $U(f,Q) \leq U(f,P)$

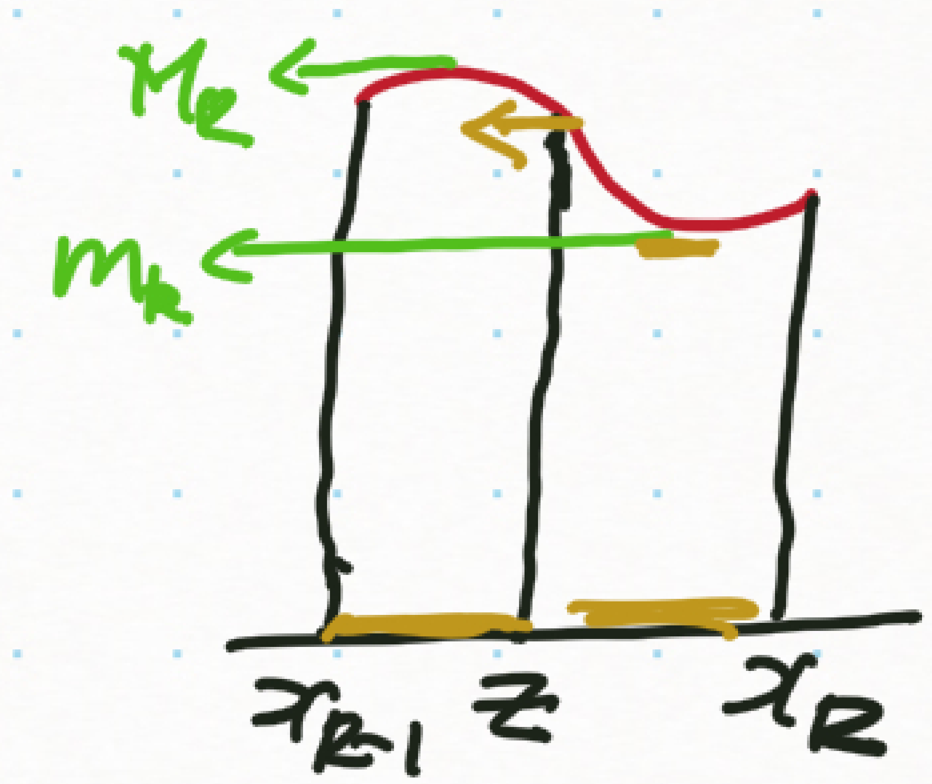
Proof Since $P \subseteq Q$ and both P & Q are finite sets, we have $|Q \setminus P|$ is finite. So we can repeat the following process $k = |Q \setminus P|$ times (or simply do induction on k):

Let $z \in Q \setminus P$ then $z \in [x_{k-1}, x_k]$ for one specific k, $1 \leq k \leq n$.

In Lower sum: $m_k(x_k - x_{k-1}) = m_k(x_k - z) + m_k(z - x_{k-1})$

$$\leq m_k'(x_k - z) + m_k''(z - x_{k-1})$$

where $m_k' = \inf\{f(x) : x \in [z, x_k]\}$
$m_k'' = \inf\{f(x) : x \in [x_{k-1}, z]\}$ $\Big\}$ each is $\geq m_k$
$m_k' \geq m_k$ & $m_k'' \geq m_k$



Lemma [Lower Sums $\leq$ Upper Sums] $f:[a,b] \to \mathbb{R}$

① If P is a partition of $[a,b]$ then $L(f,P) \leq U(f,P)$

② If P_1 and P_2 are any two partitions of $[a,b]$
then $L(f,P_1) \leq U(f,P_2)$

Proof ① By definition.

② Let $Q = P_1 \cup P_2$, the common refinement of P_1 & P_2.
So $P_1 \subseteq Q$ and $P_2 \subseteq Q$.

Applying previous lemma,

$$L(f,P_1) \leq L(f,Q) \overset{①}{\leq} U(f,Q) \leq U(f,P_2)$$

$f : [a,b] \to \mathbb{R}$

As partitions get more refined, our lower estimate <u>increases</u> and our upper estimate <u>decreases</u>, i.e. $L(f,P)$ & $U(f,P)$ get closer.

<u>Defn</u> Let $\mathcal{P}$ be the collection of all possible partitions of $[a,b]$.

<u>Upper Integral of f</u>, $U(f) = \inf\{U(f,P) : P \in \mathcal{P}\}$

<u>Lower Integral of f</u>, $L(f) = \sup\{L(f,P) : P \in \mathcal{P}\}$

$$L(f,P) \xrightarrow[\text{P is refined}]{} \Big| \int_a^b f(x)\,dx \Big| \xleftarrow[\text{P is refined}]{} U(f,P)$$

$L(f)$?? $U(f)$

$f : [a,b] \to \mathbb{R}$

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<u>Defn</u> Let $f : [a,b] \to \mathbb{R}$ be a bounded function. We say f is <u>Riemann integrable</u> if $U(f) = L(f)$.

In this case we denote: $\int_a^b f = U(f) = L(f)$.

Proposition Let $f:[a,b]\to\mathbb{R}$ be a bounded function, say $m\le f(x)\le M\ \forall x\in[a,b]$. Then

$$m(b-a)\le L(f)\le U(f)\le M(b-a)$$

Proof $L(f)=\sup\{L(f,P):P\in\mathcal{P}\}\le\inf\{U(f,P):P\in\mathcal{P}\}=U(f)$

Why? We know $L(f,P)\le U(f,P')$ for all $P,P'\in\mathcal{P}$

Show: If $x\in A$ and $y\in B\Rightarrow x\le y$ then $\sup A\le\inf B$

TRY IT!

Proposition Let $f:[a,b]\to\mathbb{R}$ be a bounded function, say $m\le f(x)\le M$ $\forall x\in[a,b]$. Then

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Show: If $x\in A$ and $y\in B \Rightarrow x\le y$ then $\sup A\le\inf B$

Let $P_0=\{a,b\}$ be the 2-point partition of $[a,b]$ ($n=1$ & $x_0=a$, $x_1=b$)

$$L(f)=\sup\{L(f,P):P\in\mathcal{P}\}$$
$$\ge L(f,P_0)$$
$$=\sum_{i=1}^{n} m_i(x_i-x_{i-1})$$
$$=m_1(b-a)\ge m(b-a)$$

$$U(f)=\inf\{U(f,P):P\in\mathcal{P}\}$$
$$\le U(f,P_0)$$
$$=\sum_{1}^{n} M_i(x_i-x_{i-1})$$
$$=M_1(b-a)\le M(b-a)$$

Examples

① $f:[a,b]\to\mathbb{R}$ given by $f(x)=c \quad \forall x\in[a,b]$

we have $\quad m=c\leq f(x)\leq c=M$.

By previous Prop., $\quad c(b-a)\leq L(f)\leq U(f)\leq c(b-a)$

That is $L(f)=U(f)$

So $\int_a^b f(x)\,dx$ exists & equals $c(b-a)$

(2) Let $f : [0,1] \to \mathbb{R}$ be the Dirichlet Function:

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Recall: its discontinuous everywhere

② Let $f : [0,1] \to \mathbb{R}$ be the Dirichlet Function :

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$] Recall: its discontinuous everywhere

Claim f is not integrable

Proof Let P be an arbitrary partition of $[0,1]$

Each subinterval $[x_{k-1}, x_k]$ of P must contain both rationals and irrationals (by density of $\mathbb{Q}$ and $\mathbb{I}$ in $\mathbb{R}$)

so, $m_k = 0$ and $M_k = 1 \ \forall k$

$$\therefore L(f,P) = \sum_{k=1}^{n} m_k (x_k - x_{k-1}) = \sum_{k=1}^{n} 0 \, (x_k - x_{k-1}) = 0$$

$$U(f,P) = \sum M_k (x_k - x_{k-1}) = \sum 1 \, (x_k - x_{k-1}) = \cancel{x_1} - x_0 + \cancel{x_2} - \cancel{x_1} + \cancel{\cdots} + x_n - \cancel{x_{n-1}}$$
$$= x_n - x_0 = 1 - 0 = 1$$

so, $L(f) = \sup\{L(f,P) : P \in \mathcal{P}\} = \sup\{0\} = 0$

$U(f) = \inf\{U(f,P) : P \in \mathcal{P}\} = \inf\{1\} = 1$

not equal.

Math 400

Real Analysis

Part #34

Are all continuous functions integrable?
What about discontinuous functions?

We know $\overset{\sup\{L(f,P)\}}{L(f)} \leq \overset{\inf\{U(f,P)\}}{U(f)}$ always

But, f integrable $\Longleftrightarrow$ $L(f) = U(f)$, i.e. we want elements of $U(f)$ to be arbitrarily close to elements of $L(f)$

Recall $a = b \Longleftrightarrow \forall \epsilon > 0 \ |a-b| < \epsilon$

$\sup\{L(f,P)\}$ $\inf\{U(f,P)\}$

We know $L(f) \leq U(f)$ always

But, f integrable $\Longleftrightarrow$ $L(f) = U(f)$, i.e. we want elements of $U(f)$ to be arbitrarily close to elements of $L(f)$

Recall: $a = b \Longleftrightarrow \forall \epsilon > 0 \ |a-b| < \epsilon$

Theorem [Integrability Criterion]

Let $f : [a,b] \to \mathbb{R}$ be bounded.

f is integrable $\Longleftrightarrow$ $\forall \epsilon > 0 \ \exists$ partition P_ϵ of $[a,b]$ such that $U(f, P_\epsilon) - L(f, P_\epsilon) < \epsilon$

$$\overset{\sup\{L(f,P)\}}{L(f)} \le \overset{\inf\{U(f,P)\}}{U(f)}$$

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Let $f : [a,b] \to \mathbb{R}$ be bounded.

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Corollary Let $f : [a,b] \to \mathbb{R}$ be bounded

f is integrable $\iff$ $\exists$ sequence of partitions P_n of $[a,b]$ s.t. $\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0$

[See Exercise 7.2.3]

We know $\overset{\sup\{L(f,P)\}}{L(f)} \leq \overset{\inf\{U(f,P)\}}{U(f)}$ always

But, f integrable $\iff$ $L(f) = U(f)$, i.e. we want elements of $U(f)$ to be arbitrarily close to elements of $L(f)$

Recall: $a = b \iff \forall \epsilon > 0 \ |a-b| < \epsilon$

Theorem [Integrability Criterion]

Let $f : [a,b] \to \mathbb{R}$ be bounded.

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Proof $\boxed{\Leftarrow}$

Given $\epsilon > 0$, suppose such a partition P_ϵ exists.

Then $U(f) - L(f) \leq U(f,P_\epsilon) - L(f,P_\epsilon) < \epsilon$

Then $U(f) = L(f)$ (by char. of $a=b$ above).

$\boxed{\Rightarrow}$ f is integrable, so $L(f) = U(f)$.

Since $U(f)$ is the inf, i.e. greatest lower bound of all $U(f,P)$

we know, Given $\epsilon > 0$, $\exists$ a partition P_1 s.t.

$$U(f, P_1) < \underline{U(f)} + \epsilon/2$$

Similarly, for $L(f)$ as the sup, we get a partition P_2 s.t.

$$L(f, P_2) < L(f) - \epsilon/2$$

$\boxed{\Rightarrow}$ f is integrable, so $L(f) = U(f)$.

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Similarly, for $L(f)$ as the sup, we get a partition P_2 s.t.

$$L(f, P_2) < L(f) - \epsilon/2 \quad \text{—— } (**)$$

Let $P_\epsilon = P_1 \cup P_2$, the common refinement.

$$\begin{aligned} U(f, P_\epsilon) - L(f, P_\epsilon) &\leq U(f, P_1) - L(f, P_2) \quad \text{[Since "Refinements Refine"]} \\ &< \left(U(f) + \epsilon/2\right) - \left(L(f) - \epsilon/2\right) \quad \text{→ by } (*) \,\&\, (**) \\ &= \int_a^b f(x)dx + \epsilon/2 - \int_a^b f(x)dx + \epsilon/2 \;=\; \frac{\epsilon}{2} + \frac{\epsilon}{2} \;=\; \epsilon \end{aligned}$$

Theorem [Continuous $\Rightarrow$ Integrable]

If $f : [a,b] \to \mathbb{R}$ is continuous, then f is integrable.

Proof f continous on compact set $\Rightarrow$ f is bounded and unif. cont..

Given $\epsilon > 0$, since f is unif. cont., $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \frac{\epsilon}{b-a} \quad \text{for all } |x - y| < \delta \text{ and all } x, y \in [a,b]$$

Let P_ϵ be a partition of $[a,b]$ s.t. $\Delta x_k = x_k - x_{k-1} < \delta \ \forall k$.

In each subinterval of P_ϵ, $[x_{k-1}, x_k]$, EVT tells us that

$$M_k = \sup_{[x_{k-1}, x_k]} f = f(z_k) \text{ for some } z_k \in [x_{k-1}, x_k]$$

$$m_k = \inf_{[x_{k-1}, x_k]} f = f(y_k) \text{ for some } y_k \in [x_{k-1}, x_k]$$

$\Rightarrow |z_k - y_k| < \delta$

$\Downarrow$

$\boxed{M_k - m_k = f(z_k) - f(y_k)}$

$< \epsilon / (b-a)$

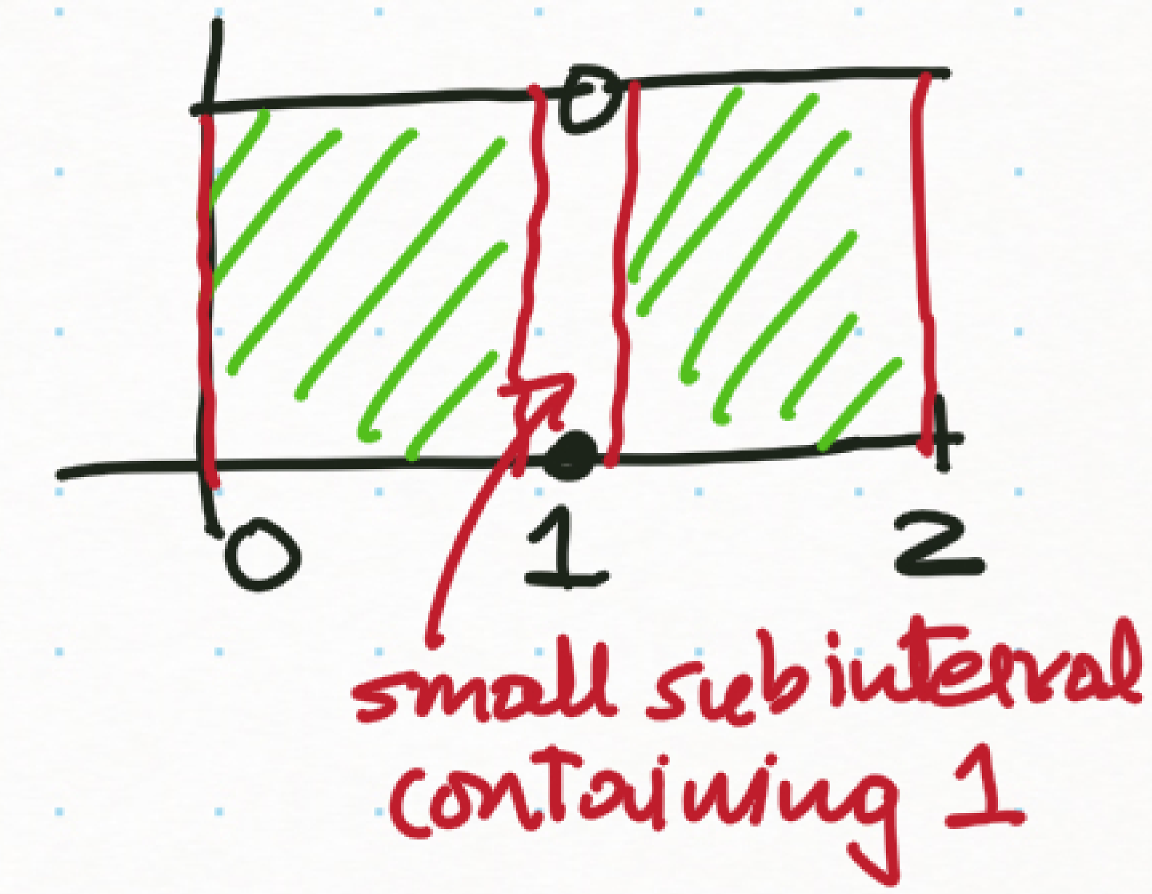
Finally, $U(f, P_\epsilon) - L(f, P_\epsilon) = \sum (M_k - m_k) \Delta x_k < \frac{\epsilon}{b-a} \sum \Delta x_k$

$$= \frac{\epsilon}{b-a}(b-a) = \epsilon$$

What about discontinuous functions?

Example

① $f(x) : [0,2] \to \mathbb{R}$ as $f(x) = \begin{cases} 1 & \text{if } x \neq 1 \\ 0 & \text{if } x = 1 \end{cases}$



Let $\underline{P_\epsilon = \{0, 1-\frac{\epsilon}{4}, 1+\epsilon/4, 2\}}$ $n=3$

In $[0, 1-\epsilon/4]$, $m_1 = 1$ and $M_2 = 1$.
In $[1-\epsilon/4, 1+\epsilon/4]$, $m_2 = 0$ and $M_2 = 1$.
In $[1+\epsilon/4, 2]$, $m_3 = 1$ and $M_3 = 1$.

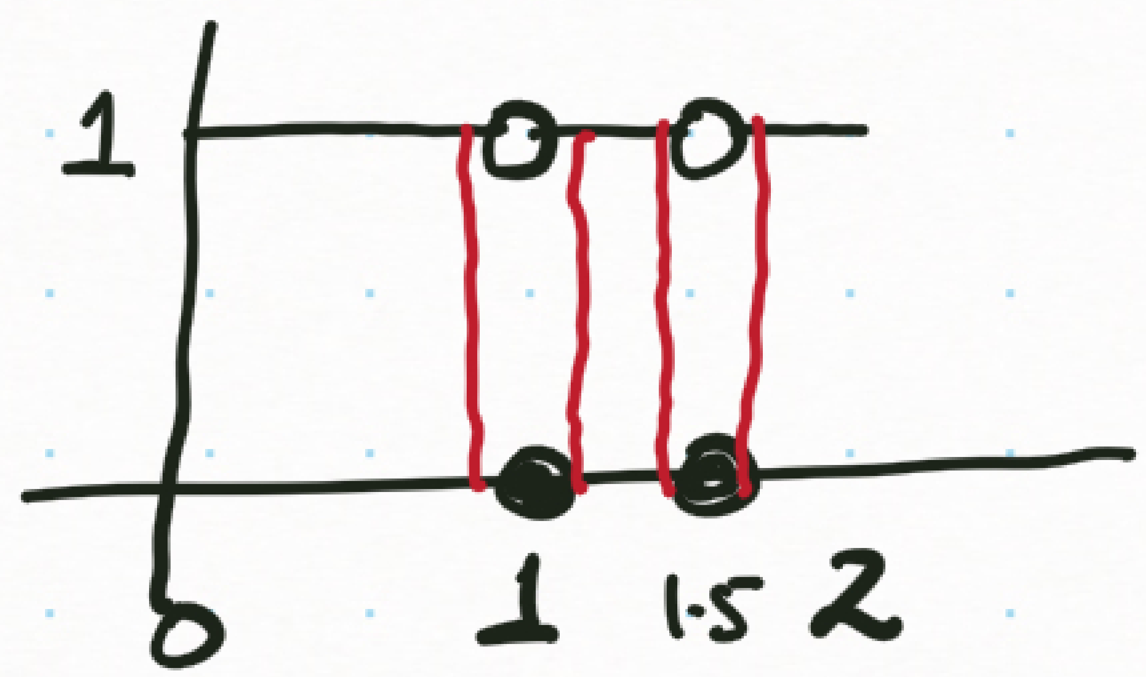
$$U(f, P_\epsilon) = \sum_{k=1}^{3} M_k (x_k - x_{k-1}) = 1\left(1-\frac{\epsilon}{4}-0\right) + 1\left(1+\frac{\epsilon}{4}-\left(1-\frac{\epsilon}{4}\right)\right) + 1\left(2-\left(1+\frac{\epsilon}{4}\right)\right)$$
$$= 1 - \frac{\epsilon}{4} + 1 + \frac{\epsilon}{4} - 1 + \frac{\epsilon}{4} + 2 - 1 - \frac{\epsilon}{4} = 2$$

$$L(f, P_\epsilon) = \sum m_k (x_k - x_{k-1}) = 1\left(1-\frac{\epsilon}{4}-0\right) + 0\left(1+\frac{\epsilon}{4}-\left(1-\frac{\epsilon}{4}\right)\right) + 1\left(2-\left(1-\frac{\epsilon}{4}\right)\right)$$
$$= 1 - \epsilon/4 + 2 - 1 - \epsilon/4 = 2 - \epsilon/2$$

$\therefore U(f, P_\epsilon) - L(f, P_\epsilon) = 2 - (2 - \epsilon/2) = \epsilon/2 < \epsilon$. $\therefore \int_0^2 f(x)\,dx$ exists & equals 2.

What if f has 2 discontinuities?

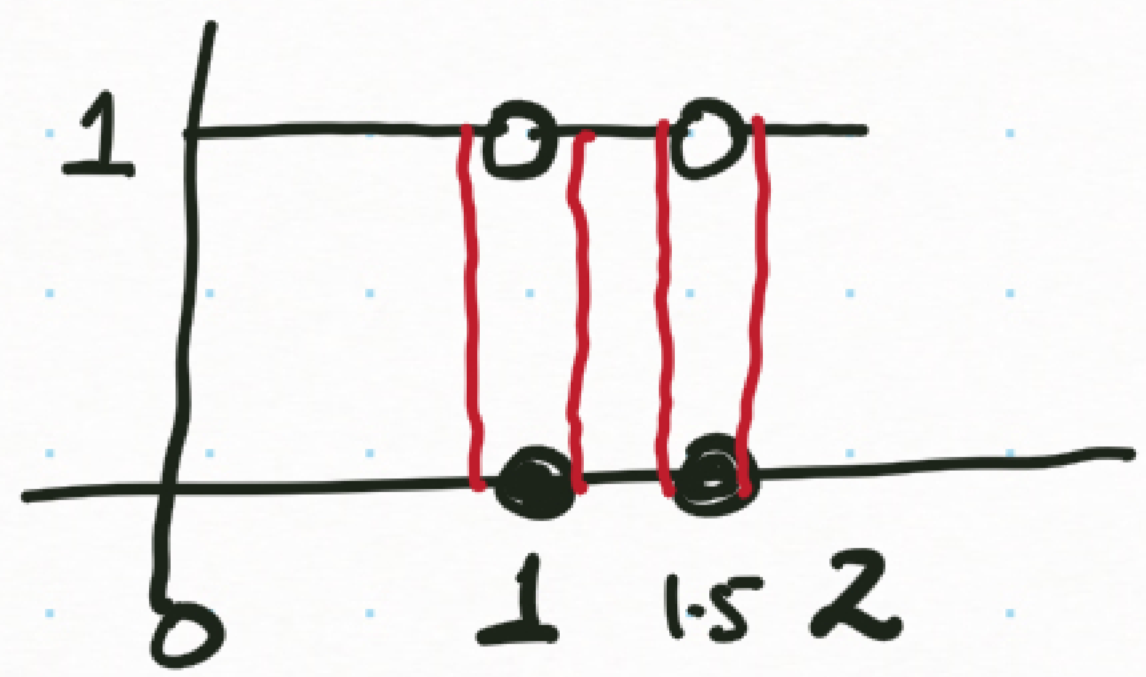
② Let $f : [0,2] \to \mathbb{R}$ as $f(x) = \begin{cases} 1 & \text{if } x \neq 1 \text{ or } 1.5 \\ 0 & \text{if } x = 1 \text{ or } 1.5 \end{cases}$



Define P_ϵ as?

What if f has 2 discontinuities?

② Let $f:[0,2]\to\mathbb{R}$ as $f(x)=\begin{cases}1 & \text{if } x\neq 1 \text{ or } 1.5\\ 0 & \text{if } x=1 \text{ or } 1.5\end{cases}$

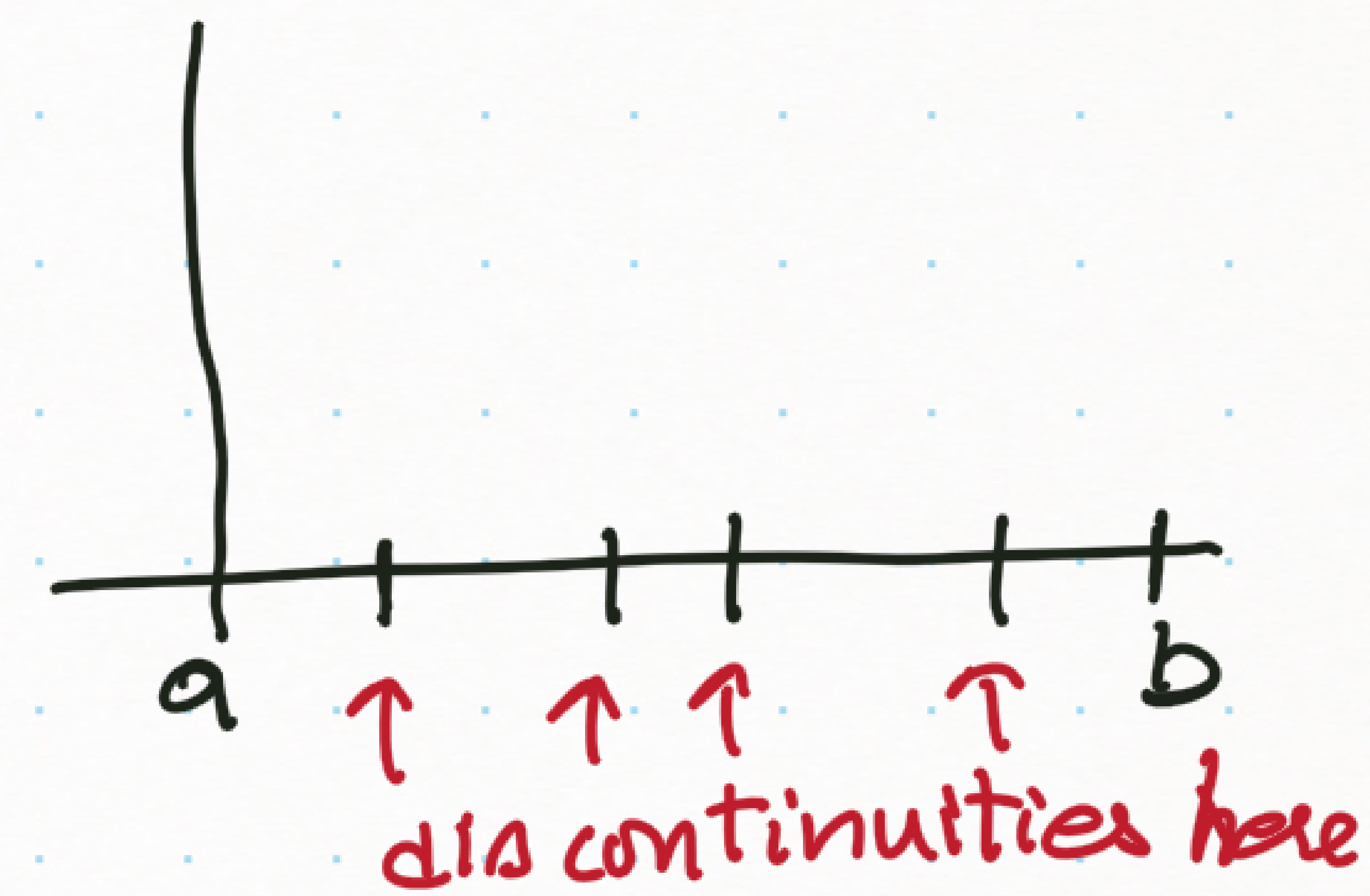


Define P_ϵ as? $P_\epsilon=\{0,\ 1-\frac{\epsilon}{8},\ 1+\frac{\epsilon}{8},\ 1.5-\frac{\epsilon}{8},\ 1.5+\frac{\epsilon}{8},\ 2\}$ $n=5$

Check (same as example ①),

$$\left.\begin{aligned} &U(f,P_\epsilon)=2\\ \text{and } &L(f,P_\epsilon)=2-\frac{\epsilon}{2}\end{aligned}\right\} \text{so } U(f,P_\epsilon)-L(f,P_\epsilon)<\epsilon$$

What if f has k discontinuities?



We could repeat the previous arguments by defining a partition P_ε that "isolates" each discontinuity.

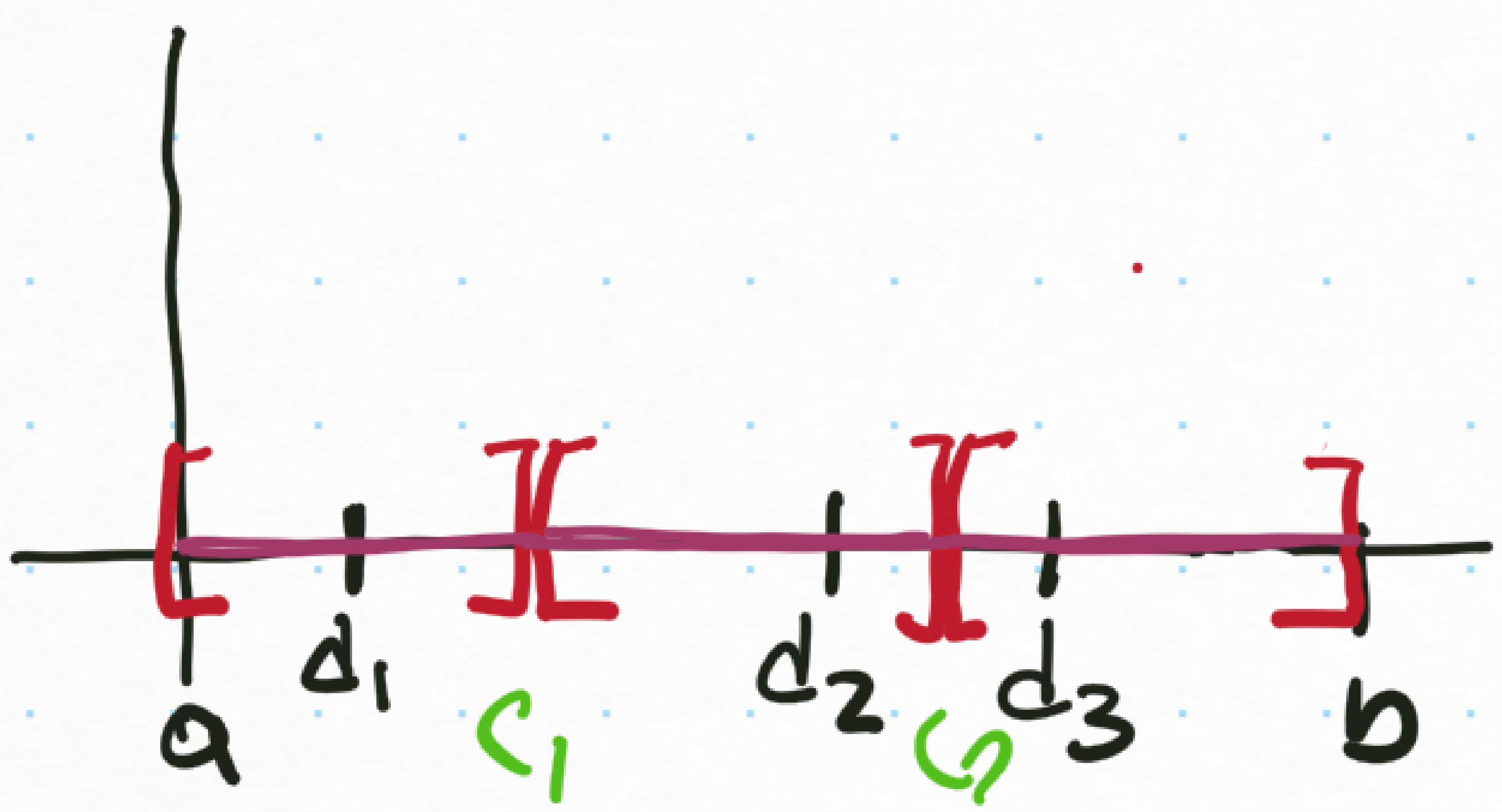
Is there an easier / cleaner way?

What if f has k discontinuities?

Using the characterization of integrability we can prove:

<u>**Theorem**</u> Let $f:[a,b]\to\mathbb{R}$ and $a<c<b$.

f is integrable on $[a,b] \iff f$ is integrable on both $[a,c]$ and $[c,b]$.



If f has discontinuities on points d_1, d_2, d_3 in $[a,b]$, then check f is integrable on 3 intervals each containing <u>exactly one</u> d_i.