Math 400: Discussion/ Review Questions # 2[1]

A statement listed with [T/F] is a True/False statement that requires a proof or a counterexample, as appropriate.

1. Let $I_n = [a_n, b_n]$ such that I_n contains I_{n+1}, for each $n \in \mathbb{N}$.
   Let $A = \{a_1, a_2, a_3, \ldots\}$. Let $B = \{b_1, b_2, b_3, \ldots\}$.
   (a) Does $\sup A$ exist? Why?
   (b) Does $\inf B$ exist? Why?
   (c) Let $a = \sup A$. Does $a \in I_n$ for all n?
   (d) Let $b = \inf B$. Does $b \in I_n$ for all n?
2. For every $n \in \mathbb{N}$, let $I_n = (0, \frac{1}{n})$. Which of the following statements are true?
   (a) $I_1 \subset I_2 \subset I_3 \subset \ldots$
   (b) $I_1 \supset I_2 \supset I_3 \supset \ldots$
   (c) $\cup_{n=1}^{m} I_n$ is nonempty for every $m \in \mathbb{N}$.
   (d) $\cap_{n=1}^{m} I_n$ is nonempty for every $m \in \mathbb{N}$.
   (e) $\cup_{n=1}^{\infty} I_n$ is nonempty.
   (f) $\cap_{n=1}^{\infty} I_n$ is nonempty.
   Why does/ does not the Nested Interval Property apply here?
3. [T/F] Given $0 < a < b$ in $\mathbb{R}$, there exists $q \in \mathbb{Z}$ such that $a < \frac{5}{q} < b$.
4. [T/F] Given $0 < a < b$ in $\mathbb{R}$, there exists $p \in \mathbb{Z}$ such that $a < \frac{p}{5} < b$.
5. [T/F] Given $0 < a < b$ in $\mathbb{R}$, there exists $p, q \in \mathbb{Z}$ such that $a < \frac{p}{q} < b$.
6. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\}$. Show that $\inf A = 0$.
7. [T/F] Let $A = \{e, \pi, \sqrt{2}\}$. Then $P(A) = \{\{e\}, \{\pi\}, \{\sqrt{2}\}, \{e, \pi\}, \{e, \sqrt{2}\}, \{\pi, \sqrt{2}\}\}$.
8. [T/F] If $A = \{1, 2, 3\}$ and $B = \{x \in \mathbb{R} : (x^2 - 1)(x^2 - 4) = 0\}$, then $A \sim B$.
9. (a) [T/F] $A \sim A$ for every set A.
   (b) [T/F] If $A \sim B$ then $B \sim A$.
   (c) [T/F] If $A \sim B$ and $B \sim C$, then $A \sim C$.
10. Apply Cantor's Diagonalization Method to the set of rational numbers in the interval $(0, 1)$. Why doesn't this show us that $\mathbb{Q} \cap (0, 1)$ is countable?
11. Some real numbers have two different decimal expansions (e.g. 0.5 is the same as 0.49999...). Why doesn't this cause any difficulties with the application of Cantor's Diagonalization Method in our proof of uncountability of $(0, 1)$?
12. Is the set of all functions from $\{0, 1\}$ to $\mathbb{N}$ countable or uncountable?

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