MATH 400: Homework #10

Only the 'Submission Problems' listed below are due Thursday, 11/9, before 10pm, via a PDF file uploaded to the Homework#10 under Assignments in the Blackboard course page.

You are allowed to discuss the homework problems with no one except your classmates, the TA, and the instructor. However, the solutions should be written by you and you alone in your own words. **If you discussed HW problems with a classmate or TA, you have to write their name at the top of the HW submission as a collaborator.** Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

Re-read the "HW Discussion and Solution Rules" and " 'Why and How' of Homework" sections of the course information sheet for some important advice on the HWs for this course.

All problems require explicit and detailed explanations. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with any HW problem, don't hesitate to ask me. You are encouraged to ask questions through the *Campuswire Discussion Forums*, during my *Office Hours*, during the *TA office hours*, or through *Email to me*.

PART I: Practice Problems

1. Try all the Discussion/ Review problems as you review the material for this week.
2. Make your own summary of this week's topics/ concepts/ properties: definitions, alternate forms of the definition based on theorems/ discussions, examples and non-examples, methods for showing the property holds or doesn't hold, useful ideas from examples, HW problems, etc.
3. Pay close attention to and attempt at least the following exercises from the textbook.
   Section 5.3: #1 (Investigation of Lipschitz functions continues; notice how Lipschitz property lies somewhere in between differentiability and uniform continuity based this and earlier problems), #3, #5, #7 (I have commented on importance of fixed points earlier; here's another little glimpse of them), #11.
   Section 6.2: #1, #3, #5, #7 (A connection between uniform continuity and uniform convergence!), #9, #11 (Dini's Theorem is a partial converse of the Limit Continuity Theorem; make a note of it).
   Section 6.3: #1, #3 (Be sure to try this!), #5.

PART II: Submission Problems

4. **Submit written solutions to the four problems listed below.**
   - Section 5.3: #3.
   - Section 6.2: #3, #9.

- Section 6.3: #1.

PART III: Readings, Comments, etc.

5. We now understand how differentiation is defined and used precisely on $\mathbb{R}$. How should we define differentiation on more general spaces?

   Lets consider a function $f : \mathbb{R}^m \to \mathbb{R}^n$. What would be a natural generalization of the definition of derivative to this setting? For a fixed non-zero $\vec{u} \in \mathbb{R}^m$ (for simplicity think of $\vec{u} = (1, 0, 0, \ldots, 0)$), we could define the derivative as $\lim_{t\to 0} \frac{f(a+\vec{u}t)-f(a)}{t}$. Intuitively what is this? Its the rate of change of f in the first coordinate (for $\vec{u} = (1, 0, 0, \ldots, 0)$). Its what is called the directional derivative (in the direction $\vec{u}$). This notion of differentiability, though appealing for its simplicity and similarity to derivatives on $\mathbb{R}$, is too weak since it does not even imply continuity of f (even if we require the limit exists for all non-zero $\vec{u}$). We need a stronger notion.

   The proper generalization of differentiability looks at it as a form of a local linearization of the given function. Geometrically we understand what the last sentence means in $\mathbb{R}$. Derivative of a function at a point geometrically represents the tangent to the curve at that point. And what is a tangent? A linear function! So, locally (that is in an ϵ-neighborhood around a point) a function behaves like this linear function. Translating the definition of derivative that we learned in this course, we have that $\phi : \mathbb{R} \to \mathbb{R}$ has a derivative at point a if there exists a number λ such that $\lim_{t\to 0} \frac{\phi(a+t)-\phi(a)-\lambda t}{t} = 0$. We then call this number λ as the derivative of the function ϕ. And the limit definition tells us that the linear function λt is a good approximation to $\phi(a + t) - \phi(a)$ (or, the tangent line $\phi(a) + \lambda t$ is a good approximation to $\phi(a + t)$) in a small neighborhood of a.

   Keeping this in mind, consider the following definition for derivative of $f : \mathbb{R}^m \to \mathbb{R}^n$. We say f is differentiable at $a \in \mathbb{R}^m$ if there exists a linear transformation $\lambda : \mathbb{R}^m \to \mathbb{R}^n$ such that $\lim_{h\to 0} \frac{f(a+h)-f(a)-\lambda(h)}{||h||} = 0$. $||h||$ indicates the norm (length from linear algebra, Math 332) of h. And speaking of linear algebra, what does a linear transformation λ look like? Its simply multiplication by a matrix! So our definition becomes f is differentiable at $a \in \mathbb{R}^m$ if there exists a $n \times m$ matrix $[\lambda]$ such that $\lim_{h\to 0} \frac{f(a+h)-f(a)-[\lambda]h}{||h||} = 0$. That is, the infinitesimal change in value of f around a is well approximated by a linear function (a matrix!). This is where the notion of Jacobian matrix (from Vector Calculus, Math 251) originates.

   This definition can be easily generalized to more general spaces beyond Euclidean spaces (Banach spaces and such). An important perspective to understanding what makes Calculus work is making the essential ideas we have learned in our coursework work in more general contexts. Manifolds, think of smooth surfaces in $\mathbb{R}^n$, are the right kind of spaces to study calculus in its most general form. In an Euclidean manifold, a neighborhood around each point looks like a Euclidean space, so we can do Calculus locally. A special kind of manifolds called Differentiable Manifolds allow us to do this local calculus in a consistent manner globally throughout the manifold. You will learn some of these ideas in a graduate course in Analysis (such as Math 500 here). To get a quick introduction, look up Michael Spivak's *Calculus on Manifolds*. Even though its a very terse book and not what I would call a good textbook, it gets to the core of the main ideas very quickly. I found it fascinating and educational when I read it as an undergrad. You can supplement it with a proper textbook like Munkres' *Analysis on Manifolds*, or a more standard textbook like Apostol's *Mathematical Analysis*.