MATH 400: Homework #8

Only the 'Submission Problems' listed below are due Thursday, 10/26, before 10pm, via a PDF file uploaded to the Homework#8 under Assignments in the Blackboard course page.

You are allowed to discuss the homework problems with no one except your classmates, the TA, and the instructor. However, the solutions should be written by you and you alone in your own words. **If you discussed HW problems with a classmate or TA, you have to write their name at the top of the HW submission as a collaborator.** Any incident of plagiarism/ cheating (from a person or from any online resource) will be strictly dealt with.

Re-read the "HW Discussion and Solution Rules" and " 'Why and How' of Homework" sections of the course information sheet for some important advice on the HWs for this course.

All problems require explicit and detailed explanations. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with any HW problem, don't hesitate to ask me. You are encouraged to ask questions through the *Campuswire Discussion Forums*, during my *Office Hours*, during the *TA office hours*, or through *Email to me*.

PART I: Practice Problems

1. Try all the Discussion/ Review problems as you review the material for this week.
2. Make your own summary of this week's topics/ concepts/ properties: definitions, alternate forms of the definition based on theorems/ discussions, examples and non-examples, methods for showing the property holds or doesn't hold, useful ideas from examples, HW problems, etc.
3. Pay close attention to and attempt at least the following exercises from the textbook.
   Section 4.3: #1, #3, #5, #7, #9, #10, #11 (Make a note of this beautiful theorem that describes a very general class of functions which have a fixed point), #13.
   Section 4.5: #2, #3, #5, #7 (Be sure to try this very short problem that illustrates the relation between solutions and fixed points).

PART II: Submission Problems

4. **Submit written solutions to 5 out of the 6 problems listed below.**
   - Section 4.3: #3, #7bc, .
   - Section 4.5: #3.
   - [Special Problem A]
     Prove that a polynomial of odd degree, that is $f(x) = \sum_{k=0}^{n} a_k x^k$ with n odd and $a_n \neq 0$, has at least one real root. What about polynomials of even degree?

- [Special Problem B]
  Prove that every positive real number has a positive nth root. Stated more formally: for each $n \in \mathbb{N}$ and $a \in (0, \infty)$, prove that there exists a $c > 0$ such that $c^n = a$.
- [Special Problem C]
  (a) Let f be a continuous real-valued function with domain (a, b). Show that if $f(r) = 0$ for each rational number r in (a, b), then $f(x) = 0$ for all $x \in (a, b)$.
  (b) Let f and g be continuous real-valued functions on (a, b) such that $f(r) = g(r)$ for each rational number r in (a, b). Prove $f(x) = g(x)$ for all $x \in (a, b)$.

PART III: Readings, Comments, etc.

5. It is possible to define continuous function purely in terms of their behavior preserving (topological) properties of sets. But, instead of describing the images of sets under a continuous function (as we did in our discussion this week), we have to consider the behavior of the preimages of sets (i.e., how sets are transformed by the inverse of the function). Continuous functions can be characterized as those functions $f : D \to \mathbb{R}$ that have the following property: for every open set C, $f^{-1}(C) := \{x \in D : f(x) \in C\}$ equals $A \cap D$ for some open set A. That is, the preimage of an open set remains an open set in the domain. Try proving this.

   This is the definition you will encounter in the study of metric and topological spaces as it naturally generalizes in many different contexts.

6. During our discussion this week, we described functions whose sets of discontinuities (i.e., points where the function is discontinuous) are $\mathbb{R}$ (Dirichlet function), $\mathbb{R} \setminus \{0\}$ (Modified Dirichlet function), $\mathbb{Q}$ (Thomae's function), $\mathbb{Z}$ (Greatest integer function).

   Can you construct functions whose sets of discontinuities are (i) $\mathbb{Z}^c$, (ii) $(0, 1]$?

   Given $C \subset \mathbb{R}$ such that C is countable. How would you define an increasing function on $\mathbb{R}$ whose set of discontinuities is equal to C? (Hint: Use a convergent infinite series of positive terms.)

7. We have seen a function which satisfies the Intermediate Value Property (IVP) but need not be continuous (see page 139 in the textbook; also note the Exercise 4.5.3 from your submission problems above). Amazingly, there is a function that has IVP in every (!) interval (no matter how big or small) but is discontinuous.

8. Look up Exercise 4.3.11 (remarked above also). It describes what is called the 'Contraction Mapping Principle' and how that leads to the existence of a fixed point ($f(x) = x$) for such a function f. This exercise is in fact a special case of what is called Banach's Fixed Point theorem.

   I could say a lot about **Fixed Point Theorems** - their beauty and their usefulness. In fact, I gave a talk, a couple of years ago, on one of my favorite theorems: Brouwer's Fixed Point Theorem. Look up my slides below for, if nothing else, the applications of Brouwer's and the related Sperner's theorems. Such a simple principle is deeply embedded in so many disparate problems from economics and mathematics to games.

   See `http://www.math.iit.edu/~kaul/talks/IIT-MFT-Brouwer-Sperner.pdf`.

   Can you figure out how to complete the proof of Brouwer's Fixed Point Theorem (from Sperner's lemma) by applying Bolzano-Weierstrass Theorem (as mentioned towards the end of the slide)?

If enough students are interested, I will be happy to give this talk during a lunch hour (with pizza :-).