

## Some Examples (Section 4.3)

① For any integer  $a$ ,  $a^2 - a + 7$  ends in one of the digits 3, 7, or 9.

Soln You are being asked to show that  $a^2 - a + 7 \equiv 3, 7, \text{ or } 9 \pmod{10}$

Since  $a \equiv r \pmod{10}$  with  $0 \leq r \leq 9$

$$a^2 - a + 7 \equiv r^2 - r + 7 \pmod{10} \text{ with } 0 \leq r \leq 9$$

For each of the 10 possibilities,  $r = 0, 1, \dots, 9$ , evaluate & reduce  $r^2 - r + 7$  modulo 10 to show that

$$r^2 - r + 7 \equiv 3, 7, \text{ or } 9 \pmod{10}$$

② For  $n \geq 5$ ,  $\sum_{k=1}^n k!$  is not a perfect square.

Soln For  $n \geq 5$ ,  $\sum_{k=1}^n k! \equiv 1! + 2! + 3! + 4! \pmod{10}$

$$\equiv 3 \pmod{10}$$

( $\because 10 \mid k!$  for  $k \geq 5$ )

~~Any  $a \equiv r \pmod{10}$~~

$$\text{Any } a \equiv r \pmod{10}, \quad 0 \leq r \leq 9$$

$$\Rightarrow a^2 \equiv r^2 \pmod{10}, \quad 0 \leq r \leq 9$$

Now, evaluate & reduce  $r^2$  modulo 10 for each of these 10 values to see that  $a^2 \equiv 0, 1, 4, 5, 6, \text{ or } 9 \pmod{10}$

Hence  $\sum_{k=1}^n k!$  cannot be a square, since

its congruent to **3** mod 10 (not 0, 1, 4, 5, 6, or 9)

③ For prime  $p > 3$ ,  $13 \mid 10^{2p} - 10^p + 1$

Soln Any prime  $p > 3$  looks like either  $6k+1$  or  $6k+5$

$$p = 6k+1 \Rightarrow 10^{2p} - 10^p + 1 \equiv (10^6)^{2k} 10^2 - (10^6)^k 10 + 1$$

$$\text{(since } 10^6 \equiv 1 \pmod{13}) \quad \equiv (1)^{2k} 100 - (1)^k 10 + 1$$

$$\equiv 100 - 10 + 1 \equiv 91 \equiv 0 \pmod{13}$$

Similarly for  $p = 6k+5$ .