Some Examples (Section 4.2)

① If $a \equiv b \pmod{n}$ then prove that $\gcd(a,n) = \gcd(b,n)$

Soln $a \equiv b \pmod{n} \Rightarrow a = b + kn$ for some k.

By Lemma in §2.3, $\gcd(a,n) = \gcd(b+kn, n) = \gcd(b,n)$

② What is $53^{103} + 103^{53}$ congruent to modulo 39

Soln Note $39 = 3 \cdot 13$

& $53 \equiv -1 \bmod 3$ & $103 \equiv 1 \bmod 3$

$\Rightarrow 53^{103} + 103^{53} \equiv (-1)^{103} + (1)^{53} \equiv 0 \pmod{3}$

$53 \equiv 1 \bmod 13$ & $103 \equiv -1 \bmod 13$

$\Rightarrow 53^{103} + 103^{53} \equiv (+1)^{103} + (-1)^{53} \equiv 0 \pmod{13}$

Since both 3 & 13 divide $53^{103} + 103^{53}$
$\gcd(3,13) = 1$ implies that 39 also divides this sum.

③ Show that $13 \mid 3^{n+2} + 4^{2n+1}$ for $n \geq 1$.
i.e., $3^{n+2} + 4^{2n+1} \equiv 0 \pmod{13}$

Soln By induction on n,

Check $n=1$

Assume $13 \mid 3^{k+2} + 4^{2k+1}$ i.e. $3^{k+2} + 4^{2k+1} \equiv 0 \bmod 13$
$(\Rightarrow 4^{2k+1} \equiv -3^{k+2} \bmod 13)$

To Show: $13 \mid 3^{k+3} + 4^{2(k+1)+1}$

$$
\begin{aligned}
3^{k+3} + 4^{2k+3} &\equiv 3(3^{k+2}) + 16(4^{2k+1}) \\
&\equiv 3(3^{k+2}) + 16(-3^{k+2}) \pmod{13} \\
&\equiv -13(3^{k+2}) \pmod{13} \\
&\equiv 0 \pmod{13}
\end{aligned}
$$

done.

④ If $a_1, \ldots, a_n$ are complete set of residues modulo n & $\gcd(a,n) = 1$, then $aa_1, \ldots, aa_n$ is also a complete set of residues modulo n.

Soln If not, then $aa_i \equiv aa_j \pmod{n}$ for $i \neq j$

$\Rightarrow a_i \equiv a_j \pmod{n}$ for $i \neq j$ ($\because \gcd(a,n) = 1$)

Contradiction, $\because$ this means $a_1, \ldots, a_n$ are not a complete set of residues