Some Examples (Section 6.1)

1. Example for proof of Thm. 6.2 (b)

\[ n = 180 = 2^2 \cdot 3^2 \cdot 5 \]

\[ (1+2+2^2) (1+3+3^2) (1+5) \]

\[ = 1 + 2 + 3 + 2^2 + 5 + 2 \cdot 3 + 3^2 + 2 \cdot 5 + 2^2 \cdot 3 + 

\[ + 3 \cdot 5 + 2 \cdot 3^2 + 2^2 \cdot 5 + 2 \cdot 3 \cdot 5 + 2^2 \cdot 3^2 + 

\[ 3^2 \cdot 5 + 2^2 \cdot 3^2 \cdot 5 \]

\[ = 1 + 2 + 3 + 4 + 5 + 6 + 9 + 10 + 12 + 15 + 18 + 20 + 30 + 36 \]

\[ + 45 + 60 + 90 + 180 = 546 \]

by Thm. 8 (a) \[ \frac{(2^3-1)}{(2^3-1)} \left( \frac{5^2-1}{5-1} \right) = 7 \cdot 13 \cdot 6 = 546 \]

These are all the divisors of 180

2. Let \( n \) be square-free number.

Then \( n = p_1 p_2 \ldots p_k \) for distinct primes \( p_i \).

Thus \( \sigma(n) = (1+1)(1+1) \ldots (1+1) = 2^k \), by Thm.

3. If \( n = 2^{k-1} \) then \( \sigma(n) = 2^n - 1 \)

\[ \sigma(n) = \sigma(2^{k-1}) = 2^k - 1 = 2 \cdot 2^{k-1} - 1 = 2n - 1 \]

by Thm.

4. For a prime number \( p \), \( \sigma(p) = 1 + p \), by Thm.

5. If \( p \neq p+2 \) are twin primes, then \( \sigma(p+2) = \sigma(p) + 2 \)

\[ \sigma(p+2) = 1 + (p+2) = (1+p)+2 = \sigma(p)+2 \]

since \( p+2 \) is prime

6. \( \sigma(n) = n^\lambda \) is a multiplicative function

\[ \sigma(nm) = (nm)^\lambda = n^\lambda m^\lambda = \sigma(n) \sigma(m) \quad \text{(need } \gcd(n,m) = 1 \text{)} \]

7. If two multiplicative functions agree on powers of primes, then they agree on all integers.

For \( p \) \text{ prime factors of } \sigma(n) = \sigma(p_1^{k_1} \ldots p_r^{k_r}) = \sigma(p_1^{k_1}) \sigma(p_2^{k_2}) \ldots \sigma(p_r^{k_r}) = q(p_1) \ldots q(p_r) \]

\[ \sigma(p_1^{k_1} \ldots p_r^{k_r}) = q(p_1^{k_1}) \ldots q(p_r^{k_r}) = q(n) \]