Illustration of proof of Theorem 7.2

Let \( m = 4 \) and \( n = 9 \). Note \( \phi(m) = 2 \) and \( \phi(mn) = 12 \).

\[ \phi(n) = 6 \]

The array of numbers is:

\[
\begin{array}{cccc}
4 & 12 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
17 & 18 & 19 & 20 \\
21 & 22 & 23 & 24 \\
25 & 26 & 27 & 28 \\
29 & 30 & 31 & 32 \\
33 & 34 & 35 & 36 \\
\end{array}
\]

Since 1, 2, 3 are co-prime to 4, the entries in the two columns below the two columns below 1, 2, 3 are all co-prime to 4. These are marked with _ boxes.

This indicates that if \( \phi(m) = 9 \cdot 2 = 18 \) numbers are co-prime to 4:

\[
\begin{array}{cccc}
29 & 30 & 31 & 32 \\
\end{array}
\]

\( \phi(n) \phi(m) = 6 \cdot 2 = 12 \) within the two columns of entries co-prime to 4. We find numbers that are co-prime to \( n = 9 \) as well. These are marked with _ circles.

\[
\begin{array}{cccc}
25 & 26 & 27 & 28 \\
\end{array}
\]

This gives a total of 18 numbers co-prime to both \( n \) and \( m \).

\[ \phi(5040) = \phi(2^4 \cdot 3^2 \cdot 5 \cdot 7) \]
\[ = \phi(2^4) \phi(3^2) \phi(5) \phi(7) = 8 \cdot 6 \cdot 4 \cdot 6 = 1152 \]

If \( n \) is odd, then \( \phi(2n) = \phi(2) \phi(n) \) (\( \because \) 2 and \( n \) are co-prime).

\[ = 1 \cdot \phi(n) = \phi(n) \]

If \( n \) is even, then \( \phi(2n) = \phi(2^{k+1} m) = \phi(2^{k+1}) \phi(m) \) (\( \because \) 2 and \( m \) are co-prime).

\[ = 2^k \phi(m) \]

\[ = 2 \cdot 2^{k-1} \phi(m) = 2 \cdot \phi(2^k \cdot m) = 2 \phi(n) \]

If \( n \) and \( n+2 \) are twin primes, then

\[ \phi(n+2) = n+1 = (n-1)+2 = \phi(n)+2 \]

If \( p \) and \( 2p+1 \) are both odd primes, then

\[ \phi(4p+2) = \phi(2 \cdot (2p+1)) = \phi(2) \phi(2p+1) \] (\( \because \) 2 and \( 2p+1 \) are co-prime).

\[ = 1 \cdot (2p) = 2(p-1)+2 = \phi(4) \phi(p)+2 = 2(\phi(p)+2) \]