Handout for Section 8.2 (part 2)

① Read Example 8.3 that shows how to find all integers with order 6 modulo 31.

We know there are $\phi(6)=2$ such integers of order 6.
To find these integers, we will first find a primitive root of 31.
There are $\phi(\phi(31))=\phi(30)=8$ primitive roots of 31.
Finding one by trial and error: since $2^5 \equiv 1 \pmod{31}$
2 cannot be a primitive root of 31. How about 3?
We need to show $3^k \not\equiv 1 \pmod{31}$ for $k \mid \phi(31)=30$ & $k<30$
i.e., $k=1,2,3,5,6,10,15$.
& then show that $3^{30} \equiv 1 \pmod{31}$ which is true by Fermat.
It turns out 3 is a primitive root of 31.
Now, to find integers of order 6 modulo 31, we need to look in the set of integers less than 31 that are coprime to 31. By Thm. 8.4, this set is given by $3, 3^2, 3^3, \ldots, 3^{30}$.
Now, $\text{ord}_{31}(3^k) = \dfrac{\text{ord}_{31}(3)}{\gcd(k, \text{ord}_{31}(3))} = \dfrac{30}{\gcd(k,30)}$

This equals 6 iff $\gcd(k,30)=5$, i.e., $k=5$ or 25.
So,
$$3^5 \equiv \ldots \equiv 26 \pmod{31} \ \&\ 3^{25} \equiv \ldots \equiv 6 \pmod{31}$$
are the only integers having order 6 modulo 31.

② Now, use the same method to find all positive integers less than 61 that have order 4 modulo 61.
(Hint: 2 is a primitive root of 61)