MATH 554 : Homework #5

You can choose one problem between problems 3 or 4, rest of the problems are compulsory. Due Thursday, April 23rd.

All problems require explicit and detailed proofs. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with it, don't hesitate to ask. You can contact me during office hours, or through email.

1. Prove that: Let p be prime and L be a set of non-negative integers. If $\mathcal{F}$ is a p-modular L-intersecting family of subsets of $[n]$, where $|L| = s$, then $|\mathcal{F}| \leq \sum_{i=0}^{s} \binom{n}{i}$.

2. Let $n \geq 2s$, and L be a set of s non-negative integers. Prove that every L-intersecting k-uniform family of subsets of $[n]$ has size at most $\binom{n}{s}$.

3. Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be subsets of $[n]$. If $|A_i \cap B_i|$ is odd for all i and $|A_i \cap B_j|$ is even for $i < j$, then $m \leq n$, and this is sharp.

4. Let $q = p^k$, where p is prime. If $A_1, \ldots, A_m$ are subsets of $[n]$ such that each $|A_i|$ is not divisible by q, but each $|A_i \cap A_j|$ is divisible by q, then $m \leq n$.

5. For $n, p \in \mathbb{N}$ with p prime and $n > 2p$, if $G_{n,p}$ is the graph whose vertices are the incidence vectors of $(2p-1)$-sets in $[n]$, with two vertices adjacent when their Euclidean-distance in $\mathbb{R}^n$ is $\sqrt{2p}$, then $\chi(G_{n,p}) \geq \binom{n}{2p-1} / \sum_{i=0}^{p-1} \binom{n}{i}$.

6. Prove that the minimum number of hyperplanes in $\mathbb{R}^n$ that do not contain the origin but together cover all the other points in $\{0,1\}^n$ (corners of the unit hypercube) is n. [Comment: For some $a, b \in \mathbb{R}^n$, a hyperplane consists of all $x \in \mathbb{R}^n$ such that $a \cdot x = b$].

7. For each vertex v in graph G, specify $B(v) \subseteq \{1, \ldots, d_G(v)\}$, where $d_G(v)$ is the degree of v in G. Prove that if $\sum_{v \in V(G)} |B(v)| < |E(G)|$, then G has a non-trivial subgraph H such that $d_H(v) \notin B(v)$ for all $v \in V(G)$ (note: not all vertices can have degree zero in H).

8. Given an odd prime p, and integer k, $1 \leq k < p$. Consider arbitrary elements $a_1, \ldots, a_k \in \mathbb{F}_p$, and distinct elements $b_1, \ldots, b_k \in \mathbb{F}_p$. Prove that there is a permutation σ of $[k]$ such that for $1 \leq i \leq k$ the values $a_i + b_{\sigma(i)}$ are distinct modulo p.