1. An integer is called *good* if we can write \( n = a_1 + a_2 + \ldots + a_k \), where \( a_1, a_2, \ldots, a_k \) are positive integers (not necessarily distinct) satisfying \( \frac{1}{a_1} + \ldots + \frac{1}{a_k} = 1 \).

Given that the integers 33, 34, 35, ..., 73 are good, prove that every integer greater than 33 is good.

2. Prove the AM-GM inequality using induction.

(Am-GM inequality: Let \( a_1, \ldots, a_n \) be non-negative real numbers, then \( (a_1 \ldots a_n)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i \).

3. Prove that \( \gcd(f_m, f_n) = f_{\gcd(m,n)} \), where \( f_n \) denotes the \( n \)th Fibonacci number.

4. Prove that in any set of 33 distinct integers with prime factors amongst \( \{5, 7, 11, 13, 23\} \), there must be two whose product is a square.

5. Prove that there is exactly one natural number \( n \) for which \( 2^8 + 2^{11} + 2^n \) is a perfect square.

6. Derive a formula for the number of quadruples \((a, b, c, d)\) such that \( 3^r 7^s = \text{lcm}(a, b, c) = \text{lcm}(b, c, d) = \text{lcm}(c, d, a) = \text{lcm}(d, a, b) \) for some \( r, s \in \mathbb{Z}^+ \).

7. Given a set \( M \) of 1539 distinct positive integers, none with a prime factor greater than 26, prove that \( M \) contains four distinct elements whose product is the fourth power of an integer.

[Compare to Optional exercise #4.]

8. Find the set of all positive integers \( n \) with the property that the set \( \{n, n+1, n+2, n+3, n+4, n+5\} \) can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.