MATH 453 : Homework #8

Solve **three** out of the following four problems. Due Thursday, 4/1.

All problems require explicit and detailed statements, especially the ones requiring a combinatorial proof. Include enough detail in your solutions so that your explanation is convincing to someone who hasn't thought about the problem before. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with it, don't hesitate to ask. You can contact me during office hours, or through email.

Supplementary Problem 1: Describe a bijection between the congruence classes of integer triangles with perimeter k (i.e., partitions of the type: $k = a + b + c$ where $a \geq b \geq c$ and $a < b + c$) and the partitions of k that have parts of sizes 2, 3, or 4 with at least one part of size 3. [I am asking you to carefully write the bijection I discussed in class.]

Supplementary Problem 2: Let $p(n, k)$ be the number of partitions of integer n into k parts. Let $A_k(x) = \sum_{n\geq 0} p(n, k)x^n$. be the OGF for partitions of integers into k parts.

(a) Prove that $p(n, k) = p(n - 1, k - 1) + p(n - k, k)$

(b) Use part (a) to prove that prove $A_k(x) = \frac{x}{1-x^k} A_{k-1}(x)$.

(c) Solve the recurrence in part (b) to get a formula for $A_k(x)$. Give a direct combinatorial explanation of why this formula is the OGF for partitions of integers into k parts.

Supplementary Problem 3: Prove Euler's second identity:

$$\prod_{i=1}^{\infty}(1 + x^{2i}) = 1 + \sum_{k\geq 1} \frac{x^{k(k+1)}}{(1 - x^2)(1 - x^4)(1 - x^6) \dots (1 - x^{2k})}$$

Hint: Compare to and use the statement of the Euler's identity stated in class.

Supplementary Problem 4: Let a_n be the number of congruence classes of integer triangles with perimeter n (i.e., partitions of the type: $n = a + b + c$ where $a \geq b \geq c$ and $a < b + c$). Prove that a_{2k} equals the number of all partitions of k into three parts (by finding a bijection between the two types of partitions).