E.g. \[
\begin{align*}
\text{max} & \quad x_1 + 4x_2 + x_3 \\
\text{s.t.} & \quad 2x_1 + 5x_2 + x_2 + x_4 = 7 - 3\theta \\
& \quad x_1 + 3x_2 + x_3 + x_5 = 5 - 2\theta \\
& \quad x_1 \geq \theta, \quad x_2 \geq 0, \quad 0 \leq x_3 \leq 3 + 2\theta, \quad x_4, x_5 \geq 0
\end{align*}
\]

**Find a basic solution:**

(Put $\theta = 0$ & see what happens —

Apply simplex to get $[0 \ 1 \ 2 \ 0 \ 0]$ as the optimal soln. with $x_2$ & $x_3$ as basic variables)

Let $x_4, x_5$ be non-basic, so put them equal to their lower bounds & then solve for $x_2$ & $x_3$ as basic variables

- $x_1 = \theta, \quad x_4 = 0, \quad x_5 = 0$
- $5x_2 + x_3 = 7 - 5\theta \quad \Rightarrow \quad x_2 = 1 - \theta$
- $3x_2 + x_3 = 5 - 3\theta \quad \Rightarrow \quad x_3 = 2$

$[\theta \ 1 - \theta \ 2 \ 0 \ 0]$ is a family of basic solutions

Each of these is dual-feasible \( (c^T - c_B B^T A \geq 0^T) \)

(i.e., $\rho^T A \leq c^T$)

Also, this is primal-feasible iff $1 - \theta \geq 0$

\( \Rightarrow 0 \leq \theta \leq 1 \)

\( \Rightarrow \theta \leq 1 \)

Case 1: Increase $\theta$ to just above 1

Now $\theta$ is no longer primal-feasible

However, apply dual-simplex starting from $[\theta \ 1 - \theta \ 2 \ 0 \ 0]$ (dual BFS).

Then, $x_2$ leaves the basis & the resulting dual simplex iteration gives
\[(\Theta \ 0 \ 7-5\Theta \ 0 \ -2+2\Theta)\]

which is a dual BFS

This is primal feasible \(\iff\ 0 \leq 7-5\Theta \leq 3+2\Theta\) and \(-2+2\Theta \geq 0\)
\(\Rightarrow\ 1 \leq \Theta \leq 1.4\)

\(\therefore\) (2) is an optimal soln. when \(1 \leq \Theta \leq 1.4\)

Case 1a: Increase \(\Theta\) above 1.4

So (2) is no longer primal feasible, but it is dual BFS

In the dual simplex iteration, \(x_3\) exits the basis, but no var. can enter the basis,
\(\therefore\) The LP is impossible when \(\Theta > 1.4\).

Case 2: Decrease \(\Theta\) below 0.5

So, (1) is no longer primal feasible, but it is dual BFS

In the dual simplex iteration, \(x_3\) exits and we get:
\[(\Theta \ \theta - 1.4 \ 3+2\Theta \ 0 \ -0.4 \ -0.88)\]

is the new dual BFS

This is primal feasible \(\iff\ \theta - 0.4 - 0.88 \geq 0\)
\(\Rightarrow\ \theta \geq -0.5\)
\(\therefore\) (3) is an optimal soln. when \(\Theta \leq -0.5\)