# MATH 554 : Homework #1

Do five out of the following six problems. Due Monday, January 31st.

All problems require explicit and detailed proofs. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with it, don't hesitate to ask. You can contact me during office hours, or through email.

**1.** Let $G = (V, E)$ be a bipartite graph on $n$ vertices with a list $S(v)$ of at least $1 + \log_2 n$ colors associated with each vertex $v \in V$. Prove that there is a proper coloring of $G$ assigning to each vertex $v$ a color from its list $S(v)$.

**2.** The Van der Waerden number $w(l, k)$ is the least $n$ such that for every $k$-coloring of $[n]$ has a monochromatic $l$-term arithmetic progression. Prove that $w(l, k) > (lk^{l-1})^{1/2}$.

**3.** Prove that every graph $G$ with $n$ vertices and $m$ edges has a bipartite subgraph with at least $m\frac{\lceil n/2 \rceil}{2\lceil n/2 \rceil - 1}$ edges.

**4.** Prove that if a graph $G$ with $n$ vertices and $m$ edges has a matching with $k$ edges, then $G$ has a bipartite subgraph with at least $(m + k)/2$ edges.

**5.** Let $\sigma$ be a permutation of $[n]$. There is a descent at $i, 1 \leq i \leq n$, if $\sigma(i) > \sigma(i+1)$. Compute the expected number of descents in a random permutation $\in S_n$.

**6.** (a) Prove that if there exists a $p \in (0, 1)$ such that $\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}} < 1$, then $R(k, l) > n$.
(b) Prove that $R(k, l) > n - (\binom{n}{k}p^{\binom{k}{2}} + \binom{n}{l}(1-p)^{\binom{l}{2}})$ for all $n \in \mathbb{N}$ and $p \in (0, 1)$. Choose $n$ and $p$ in this bound (in terms of $l$) to prove $R(3, l) > l^{3/2-o(1)}$, where $o(1)$ represents a function of $l$ that tends to zero as $l$ tends to infinity.