MATH 554 : Homework #2

Do five out of the following six problems. Due Friday, February 11th.

All problems require explicit and detailed proofs. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with it, don't hesitate to ask. You can contact me during office hours, or through email.

1. Consider Boolean formulae, $S = C_1 \wedge C_2 \wedge \ldots \wedge C_m$, that are formed by conjunction of clauses where each clause, $C_i = a_1 \vee a_2 \vee \ldots \vee a_k$, is a disjunction of k literals involving k distinct boolean variables. A literal is either a variable or its negation. Let S be a Boolean formula of the type above such that no variable occurs in more than $2^{k-\log_2 k-2}$ clauses, then prove that S is satisfiable, i.e. there exists an assignment of True/ False to the variables such that S is true.

2. The Van der Waerden number $w(l,k)$ is the least n such that every k-coloring of $[n]$ has a monochromatic l-term arithmetic progression. Use LLL to prove that $w(l,k) > (1 + o(1)(ekl)^{-1}k^l$. (Hint: You only need to give an asymptotic upper bound on the max degree of the dependency graph.) (Comment: When l is a prime, there is a construction for $w(l,2) > l2^l$ using finite fields.)

3. The maximum size (number of edges) of an n-vertex graph not containing H is denoted $ex(n;H)$. Use the deletion method to prove that $ex(n;C_k) \geq \Omega(n^{1+1/(k-1)})$. In other words, prove that there is a constant c_k such that there exists an n-vertex graph with at least $c_k n^{1+1/(k-1)}$ edges that does not contain a k-cycle.

4. Let $\{A_i\}_{i=1}^m$ and $\{B_i\}_{i=1}^m$ be subsets of [n], with $|A_i| = a_i$ and $|B_i| = b_i$, and $A_i \cap B_j = \emptyset$ if and only if $i = j$. Prove that $\sum_{i=1}^m \binom{a_i+b_i}{a_i}^{-1} \leq 1$. Apply this to prove that the maximum size of a family of pairwise incomparable (neither is a subset of the other) subsets of $[n]$ is $\binom{n}{\lfloor n/2 \rfloor}$. (Hint: Consider the probability space of permutations of $[n]$, equally likely, and define an appropriate event.)

5. Let G be a digraph in which every vertex has an outdegree k and indegree k. Let $r = \lfloor k/(2.25 + 2\log k) \rfloor$. Partition $V(G)$ into r sets $V_1, \ldots, V_r$ by placing each vertex, independently, into a random V_i, chosen uniformly. Use LLL to prove that with positive probability every vertex has a successor in the set containing it. Conclude that every k-regular directed graph has a family of r pairwise disjoint cycles. (Comment: It is conjectured that $k/2$ disjoint cycles can be found, and $k/64$ is known.)

6. For $k \geq 9$, every k-regular k-uniform hypergraph has a proper 2-coloring.