MATH 554 : Homework #4

Do all of the following five problems. Due Monday, March 21th. All problems should be solved using Entropy.

All problems require explicit and detailed proofs. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with it, don't hesitate to ask. You can contact me during office hours, or through email.

1. In class we used Entropy to prove that $n^2 \leq n_1 n_2 n_3$ where n is the number of distinct points in $\mathbb{R}^3$ which have n_1, n_2, n_3 distinct projections on the three standard planes, respectively. State and prove a generalization of this result for points in $\mathbb{R}^d$. [Comment: Many different generalizations are possible based on which projections you wish to consider.]

2. Define $[n] = \{0, \ldots, n-1\}$. Fix $k \in [n]$. Let $\mathcal{F} \subseteq 2^{[n]}$ such that $\forall F, F' \in \mathcal{F}$, $\exists i \in [n]$ such that $\{i + j \pmod n \mid 0 \leq j \leq k\} \subseteq F \cap F'$. Then state and prove a sharp upper bound on $|\mathcal{F}|$.

3. (a) Let $G = (A, B; E)$ be a bipartite graph with $|A| = |B| = n$. Then prove that the number of perfect matchings in G is at most $\prod_{v \in A} d(v)$, where $d(v)$ is the degree of vertex v.

(b) The famous Bregman's bound states that the number of perfect matchings in G is at most $\prod_{v \in A} (d(v)!)^{1/d(v)}$. As an optional problem try proving this. Which part of your proof of (a) needs to be improved to get Bregman's bound? There is a short Entropy based proof of Bregman's bound, ask me for the reference.

4.(a) Suppose $G_1, \ldots, G_t$ be bipartite graphs with the same vertex set $[n]$ such that union of their edge sets equals K_n. The prove that $t \geq \log n$. [Comment: A short proof using properties of chromatic number is possible but I am looking for an entropy based proof.]

(b) State and prove a generalization where G_i are all k-partite graphs. [Comment: Fredman and Komlos generalize this result to decomposition of a complete r-uniform hypergraph on n vertices into t k-partite r-uniform hypergraphs.]

5. Let k be a positive integer. Let Ω be a finite set and $\mathcal{S} = \{S_1, \ldots, S_m\}$ be a collection of subsets of Ω such that each element of Ω is contained in at least k members of $\mathcal{S}$. Let $\mathcal{F}$ be a collection of subsets of Ω. For each $f \in \mathcal{F}$, denote $f \cap S_i$ by f_i and let $\mathcal{F}_i = \{f_i | f \in \mathcal{F}\}$. Let every such set f_i be endowed with a positive integral weight $w_i(f_i)$. Then prove that

$$\left(\sum_{f \in \mathcal{F}} \prod_{i=1}^{m} w_i(f_i) \right)^k \leq \prod_{i=1}^{m} \sum_{f_i \in \mathcal{F}_i} (w_i(f_i))^k$$

[Note that setting all the weights equal to 1 gives us the corollary to Shearer's lemma that we proved in class. An easy generalization of this result can be used to prove classical inequalities like Cauchy-Schwarz, Holder, and many more.]