Example

$$\min \; -x_1 + 2x_2 + 3x_3 + x_4 + x_5 - 2x_6$$

s.t.

$$x_1 + 2x_2 + 2x_3 + x_4 + x_5 = 12$$
$$x_1 + 2x_2 + x_3 + x_4 + 2x_5 + x_6 = 18$$
$$3x + 6x_2 + 2x_3 + x_4 + 3x_5 = 24$$
$$x \geq 0$$

In std. form. How to find initial BFS?

Phase I

$$\min \; y_1 + y_2 + y_3$$

s.t.

$$x_1 + 2x_2 + 2x_3 + x_4 + x_5 + y_1 = 12$$
$$x_1 + 2x_2 + x_3 + x_4 + 2x_5 + x_6 + y_2 = 18$$
$$3x_1 + 6x_2 + 2x_3 + x_4 + 3x_5 + y_3 = 24$$
$$x \geq 0, \; y \geq 0$$

Initial BFS is $x_1 = x_2 = x_3 = \cdots = x_6 = 0$

$y_1 = 12 \quad y_2 = 18 \quad y_3 = 24$

with $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

		↓								
		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3
	-54	-5	-10	-5	-3	-6	-1	0	0	0
$y_1 =$	12	1	2	2	1	1	0	1	0	0
$y_2 =$	18	1	2	1	1	2	1	0	1	0
no choice ← $y_3 =$	24	3	6	2	1	3	0	0	0	1

				↓						
		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3
	-14	0	0	$-5/3$	$-4/3$	-1	-1	0	0	$5/3$
no choice ← $y_1 =$	4	0	0	$4/3$	$2/3$	0	0	1	0	$-1/3$
$y_2 =$	10	0	0	$1/3$	$2/3$	1	1	0	1	$-1/3$
$x_2 =$	4	$1/2$	1	$1/3$	$1/6$	$1/2$	0	0	0	$1/6$

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3
	-9	0	0	0	$-1/2$	-1	-1	$5/4$	0	$5/4$
$x_3 =$	3	0	0	1	$1/2$	0	0	$3/4$	0	$-1/4$
no choice $y_2 =$	9	0	0	0	$1/2$	1	1	$-1/4$	1	$-1/4$
$x_2 =$	3	$1/2$	1	0	0	$1/2$	0	$-1/4$	0	$1/4$

		x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3
	0	0	0	0	0	0	0	1	1	1
$x_3 =$	3	0	0	1	$1/2$	0	0	$3/4$	0	$-1/4$
$x_6 =$	9	0	0	0	$1/2$	1	1	$-1/4$	1	$-1/4$
$x_2 =$	3	$1/2$	1	0	0	$1/2$	0	$-1/4$	0	$+1/4$

Optimality with cost = 0

All artificial variables are nonbasic variables
So, $x = [0 \ 3 \ 3 \ 0 \ 0 \ 9]^T$ is a BFS for the original LP with $B = [A_3 \ A_6 \ A_2]$ as the initial basis.

note the order of the subscripts

<u>Phase II</u>

Start with the sub-tableau from Phase I

		x_1	x_2	x_3	x_4	x_5	x_6
	3	-2	0	0	$1/2$	2	0
$x_3 =$	3	0	0	1	$1/2$	0	0
$x_6 =$	9	0	0	0	$1/2$	1	1
$x_2 =$	3	$1/2$	1	0	0	$1/2$	0

The cost & the reduced costs have to be calculated using c^T from the original LP.

		x_1	x_2	x_3	x_4	x_5	x_6
	15	0	4	0	$1/2$	4	0
$x_3 =$	3	0	0	1	$1/2$	0	0
$x_6 =$	9	0	0	0	$1/2$	1	1
$x_1 =$	6	1	2	0	0	1	0

Optimal solution $x^T = [6 \ 0 \ 3 \ 0 \ 0 \ 9]$ with cost = 15

In the LP we just solved using the Two phase method, x_6 can be used as basic variable because it occurs in only one equation with coefficient $+1$.

$$\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{bmatrix} = \vec{b}$$

a standard basis vector

So, we only need to introduce ~~vector~~ artificial variables into 1st & 3rd equations.

<u>Phase I</u>

$$\min \quad y_1 + y_2$$
s.t.
$$\begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 2 & 1 & 0 & 0 \\ 3 & 6 & 2 & 1 & 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 12 \\ 18 \\ 24 \end{bmatrix}$$

Initial BFS is

$$\begin{array}{cccccccc} & & & & & x_6 & y_1 & y_2 \\ \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 18 & 12 & 24 \end{bmatrix} \text{ with basis}$$

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

		x_1	x_2 ↓	x_3	x_4	x_5	x_6	y_1	y_2
	-36	-4	-8	-4	-2	-4	0	0	0
$y_1 =$	12	1	2	2	1	1	0	1	0
$x_6 =$	18	1	2	1	1	2	1	0	0
← $y_2 =$	24	3	6	2	1	3	0	0	1

				↓					
	-4	0	0	$-4/3$	$-2/3$	0	0	0	$4/3$
← $y_1 =$	4	0	0	$4/3$	$2/3$	0	0	1	$-1/3$
$x_6 =$	10	0	0	$1/3$	$2/3$	1	1	0	$-1/3$
$x_2 =$	4	$1/2$	1	$1/3$	$1/6$	$1/2$	0	0	$1/6$

	0	0	0	0	0	0	0	1	1
$x_3 =$	3	0	0	1	1/2	0	0	3/4	-1/4
$x_6 =$	9	0	0	0	1/2	1	1	-1/4	-1/4
$x_2 =$	3	1/2	1	0	0	1/2	0	-1/4	1/4

$\therefore$ initial BFS for phase II is

$$x = [0 \ 3 \ 3 \ 0 \ 0 \ 9] \text{ with}$$

$$\text{basis } B = [A_3 \ A_6 \ A_2]$$

<u>example</u> min $-x_1 - 2x_2 - x_3$

s.t. $3x_1 + x_2 - x_3 = 15$

$8x_1 + 4x_2 - x_3 = 50$

$2x_1 + 2x_2 + x_3 = 20$

$\vec{x} \geq 0$

<u>Phase I</u>

We introduce artificial vars y_1, y_2, y_3 To get

		$x_1 \downarrow$	x_2	x_3	y_1	y_2	y_3
	-85	-13	-7	1	0	0	0
$\leftarrow y_1 =$	15	3	1	-1	1	0	0
$y_2 =$	50	8	4	-1	0	1	0
$y_3 =$	20	2	2	1	0	0	1

$\vdots$ at the end of Phase I we get

	0	0	0	0	3	0	2
$x_1 =$	7	1	3/5	0	1/5	0	1/5
$y_2 =$	0	0	0	0	-2	1	-1
$x_3 =$	6	0	4/5	1	-2/5	0	3/5

The optimal soln. includes artificial var $y_2 = 0$
We need to drive it out of the basis.
y_2 is the <u>second</u> basic var. & the <u>second</u> entry of all the columns $B^{-1}A_j$, $j = 1, 2, 3$ are all zero.

This means the matrix A has lin. dependent rows & we can remove the second row of the Tableau which pushes y_2 out of the basis & we can begin Phase II as

		x_1	x_2	x_3	
$x_1 =$	7	1	3/5	0	
$x_3 =$	6	0	4/5	1	

Calculate the cost & reduced costs using the original LP c^T.

We start with $x = [7 \quad 0 \quad 6]$ as the BFS with basis $B = [A_1 \quad A_3]$

→ Now imagine that at the end of Phase I. the optimal Tableau looked like

		x_1	$x_2 \downarrow$	x_3	y_1	y_2	y_3
	0	0	0	0	3	0	2
$x_1 =$	7	1	3/5	0	1/5	0	1/5
$\leftarrow y_2 =$	[illegible]	0	1	0	*	*	*
$x_3 =$	6	0	4/5	1	-2/5	0	3/5

Then to drive out y_2, we note that y_2 is the <u>second</u> basic var & the <u>second</u> entry of $B^{-1}A_2$ is non-zero.

So, use row op.s to make x_2 enter the basis & y_2 leave the basis.

This will give an initial BFS for the original LP with Basis consisting only of ~~an~~ columns of A.

Now, see Example 3.8 in the book.