MATH 554 : Homework #6

Solve 3 out of the following 4 problems. Due Thursday, April 25th.

All problems require explicit and detailed proofs. Solutions should be written clearly, legibly, and concisely, and will be graded for both mathematical correctness and presentation. Points will be deducted for sloppiness, incoherent or insufficient explanation, or for lack of supporting rationale.

Always remember that homework is NOT meant to be an examination, it is meant to assist in your learning and development. If you need help with it, don't hesitate to ask. You can contact me during office hours, or through email.

1. Prove that the minimum number of hyperplanes in $\mathbb{R}^n$ that do not contain the origin but together cover all the other points in $\{0,1\}^n$ (corners of the unit hypercube) is n. [Comment: For some $a, b \in \mathbb{R}^n$, a hyperplane consists of all $x \in \mathbb{R}^n$ such that $a \cdot x = b$].

2. For each vertex v in graph G, specify $B(v) \subseteq \{1, \ldots, d_G(v)\}$, where $d_G(v)$ is the degree of v in G. Prove that if $\sum_{v \in V(G)} |B(v)| < |E(G)|$, then G has a non-trivial subgraph H such that $d_H(v) \notin B(v)$ for all $v \in V(G)$ (note: non-trivial means not all vertices can have degree zero in H).

3. Given an odd prime p, and integer k, $1 \leq k < p$. Consider arbitrary elements $a_1, \ldots, a_k \in \mathbb{F}_p$, and distinct elements $b_1, \ldots, b_k \in \mathbb{F}_p$. Prove that there is a permutation σ of $[k]$ such that for $1 \leq i \leq k$ the values $a_i + b_{\sigma(i)}$ are distinct modulo p.

4. Let p be a prime and $(a_1, b_1), \ldots, (a_{2p-1}, b_{2p-1})$ be a sequence of integers such that $a_i, b_i \in \mathbb{Z}_p$, $\forall i$. Prove that there exists a non-empty set $I \subseteq \{1, \ldots, 2p-1\}$ such that $\sum_{i \in I}(a_i, b_i) = (0, 0)$ modulo p.