Some Examples (Section 4.3)

(1) For any integer a, $a^2 - a + 7$ ends in one of the digits 3, 7, or 9.

Soln You are being asked to show that

$$a^2 - a + 7 \equiv 3, 7, \text{ or } 9 \pmod{10}$$

Since $a \equiv r \pmod{10}$ with $0 \le r \le 9$

$$a^2 - a + 7 \equiv r^2 - r + 7 \pmod{10} \text{ with } 0 \le r \le 9$$

For each of the 10 possibilities, $r = 0, 1, \ldots, 9$, evaluate & reduce $r^2 - r + 7$ modulo 10 to show that

$$r^2 - r + 7 \equiv 3, 7, \text{ or } 9 \pmod{10}$$

(2) For $n \ge 5$, $\sum_{k=1}^{n} k!$ is not a perfect square.

Soln For $n \ge 5$, $\sum_{k=1}^{n} k! \equiv 1! + 2! + 3! + 4!$
$\equiv 3 \pmod{10}$

($\because 10 \mid k!$ for $k \ge 5$)

~~Any~~

Any $a \equiv r \pmod{10}$, $0 \le r \le 9$
$\Rightarrow a^2 \equiv r^2 \pmod{10}$, $0 \le r \le 9$

Now, evaluate & reduce r^2 modulo 10 for each of these 10 values to see that

$$a^2 \equiv 0, 1, 4, 5, 6, \text{ or } 9 \pmod{10}$$

Hence $\sum_{k=1}^{n} k!$ cannot be a square, since its congruent to 3 mod 10 (not 0, 1, 4, 5, 6, or 9)

(3) For prime $p > 3$, $13 \mid 10^{2p} - 10^p + 1$

Soln Any prime $p > 3$ looks like either $6k+1$ or $6k+5$

$p = 6k+1 \Rightarrow$ $10^{2p} - 10^p + 1 \equiv (10^6)^{2k} 10^2 - (10^6)^k 10 + 1$

(since $10^6 \equiv 1 \pmod{13}$) $\equiv (1)^{2k} 100 - (1)^k 10 + 1$

$\equiv 100 - 10 + 1 \equiv 91 \equiv 0 \pmod{13}$

Similarly for $p = 6k+5$.