Handout (section 6.2)

① Illustration of Proof of MIF

$n=10$: $\displaystyle\sum_{d|10}\left(\sum_{c|(\frac{10}{d})}\mu(d)f(c)\right)$

$= \mu(1)[f(1)+f(2)+f(5)+f(10)] + \mu(2)[f(1)+f(5)] + \mu(5)[f(1)+f(2)]$
$+\mu(10)f(1)$

rearranging terms w.r.t. f

$= f(1)[\mu(1)+\mu(2)+\mu(5)+\mu(10)] + f(2)[\mu(1)+\mu(5)] + f(5)[\mu(1)+\mu(2)]$
$+ f(10)\mu(1)$

$= \displaystyle\sum_{c|10}\left(\sum_{d|(\frac{10}{c})}\mu(d)f(c)\right)$

How about quotient f/g (given $g(n)\neq 0\ \forall n$)?

② Product of two multiplicative functions is also multiplicative:

Let f & g be multiplicative. To show: fg is also multiplicative

Given m,n with $\gcd(m,n)=1$,

$(fg)(mn) = f(mn)g(mn) = f(m)f(n)g(m)g(n) = (f(m)g(m))(f(n)g(n))$
$= (fg)(m)\,(fg)(n)$ done

③ For any $n\geq 1$, $\mu(n)\mu(n+1)\mu(n+2)\mu(n+3)=0$

Proof.
Of any four consecutive integers $n, n+1, n+2, n+3$, one will be divisible by $4\ (=2^2)$, hence not sq-free, so will have μ-value of 0.

④ For $n\geq 3$, $\displaystyle\sum_{k=1}^{n}\mu(k!)=1$

Proof:
$\displaystyle\sum_{k=1}^{n}\mu(k!) = \mu(1!)+\mu(2!)+\mu(3!)+\underbrace{\mu(4!)+\dots+\mu(n!)}$

each of these is divisible by $4\ (=2^2)$ hence μ-value is 0

$= \mu(1)+\mu(2)+\mu(6)$
$= 1+(-1)+1 = 1$

Important

⑤ If f is mult., then prove that $F(n)=\sum_{d|n}\mu(d)f(d) = (1-f(p_1))(1-f(p_2))\cdots(1-f(p_r))$
where $n=p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}$ is the prime factorization of n.

Proof: Since both μ & f are mult., μf is also mult..
Applying Thm 6.4, F is also multiplicative (so its enough to find $F(p^k)$, p prime)

$F(p^k) = \mu(1)f(1)+\mu(p)f(p)+\dots+\mu(p^k)f(p^k) = \mu(1)f(1)+\mu(p)f(p)$

(all other μ-values are 0)

$= 1-f(p)$ ⊛ ($\because f(1)=1$ as a mult. ftn.)

So, $F(n) = F(p_1^{k_1})F(p_2^{k_2})\cdots F(p_r^{k_r})$

using ⊛ $= (1-f(p_1))(1-f(p_2))\cdots(1-f(p_r))$ done.