Handout (Section 6.2)

1) Illustration of Rec. of MIF
\[ n=10; \quad \sum_{d|10} \left( \sum_{e|d} \mu(d)s(e) \right) \]
\[ = \mu(4)\left[ s(1)+s(2)+s(5)+s(10) \right] + \mu(2)\left[ s(1)+s(5) \right] + \mu(5)s(2) \]
\[ + \mu(10)s(1) \]
\[ = s(4)\left[ \mu(1)+\mu(2)+\mu(5)+\mu(10) \right] + s(2)\left[ \mu(1)+\mu(5) \right] + s(5)\left[ \mu(1)+\mu(2) \right] \]
\[ + s(10)\mu(1) \]
\[ = \sum_{d|10} \sum_{e|d} \mu(d)s(e) \]

How about quotient \( \frac{s_q}{g} \) (given \( g|n \?))

2) Product of two multiplicative functions is also multiplicative:
Let \( f, g \) be multiplicative. To show: \( fg \) is also multiplicative.
\[ \gcd(m, n) = 1 \]
\[ (fg)(mn) = f(m)g(m)g(n) = \frac{f(m)g(m)f(n)g(n)}{(fg)(m)(fg)(n)} \]

3) For any \( n \geq 1 \), \( \mu(n)\mu(n+1)\mu(n+2)\mu(n+3) = 0 \)
\[ \text{Proof:} \]
If any four consecutive integers \( n, n+1, n+2, n+3 \), one will be divisible by 4 (=2^2), hence not squarefree, so will have \( \mu \)-value \( 0 \).

4) For \( n \geq 3 \), \( \sum_{k=1}^{n} \mu(k) = 1 \)
\[ \sum_{k=1}^{n} \mu(k) \]
\[ = \mu(1) + \mu(2) + \mu(3) + \mu(4) + \ldots + \mu(n) \]
\[ = \mu(1) + \mu(2) + \mu(6) \]
\[ = 1 + (-1) + 1 = 1 \]

Important

5) If \( f \) is mult., then prove that \( F(n) = \sum_{d|n} \mu(d)s(d) = (1-s(p_1))(1-s(p_2)) \ldots (1-s(p_k)) \)
where \( n = p_1^{a_1}p_2^{a_2} \ldots p_k^{a_k} \) is the prime factorization of \( n \).
\[ \text{Proof:} \]
Since both \( \mu \) & \( s \) are mult., \( \mu \cdot s \) is also mult.
Applying Thm 6.4, \( F \) is also multiplicative (so its enough to find \( F(p^k) \), \( p \) prime)
\[ F(p^k) = \mu(1)s(1) + \mu(p)s(p) + \ldots + \mu(p^k)s(p^k) = \mu(1)s(1) + \mu(p)s(p) \]
\[ = 1 - s(p) \quad (\because \mu(1) = 1 \text{ as a mult. fun.}) \]
\[ \text{all other \( \mu \)-values are 0} \]
\[ \Rightarrow F(n) = F(p_1^{a_1})F(p_2^{a_2}) \ldots F(p_k^{a_k}) \]
using \( \otimes = (1-s(p_1))(1-s(p_2)) \ldots (1-s(p_k)) \) done.