MATH 410 : Optional Exercises

This file containing challenging problems from outside the textbook will be updated throughout the semester. These problems should be attempted only if you have finished all the homework problems, or if you need a problem more challenging than the HW problems.

1. An integer is called *good* if we can write $n = a_1 + a_2 + \ldots + a_k$, where $a_1, a_2, \ldots, a_k$ are positive integers (not necessarily distinct) satisfying $\frac{1}{a_1} + \ldots + \frac{1}{a_k} = 1$.
Given that the integers 33, 34, 35, ..., 73 are good, prove that every integer greater than 33 is good.

2. Prove the AM-GM inequality using induction.
(AM-GM inequality: Let $a_1, \ldots, a_n$ be non-negative real numbers, then $(a_1 \ldots a_n)^{1/n} \leq \frac{1}{n} \sum_{i=1}^{n} a_i$.)

3. Prove that $gcd(f_m, f_n) = f_{gcd(m,n)}$, where f_n denotes the nth Fibonacci number.

4. Prove that in any set of 33 distinct integers with prime factors amongst $\{5, 7, 11, 13, 23\}$, there must be two whose product is a square.

5. Prove that there is exactly one natural number n for which $2^8 + 2^{11} + 2^n$ is a perfect square.

6. Derive a formula for the number of quadruples (a, b, c, d) such that $3^r 7^s = lcm(a, b, c) = lcm(b, c, d) = lcm(c, d, a) = lcm(d, a, b)$ for some $r, s \in \mathbb{Z}^+$.

7. Given a set M of 1539 distinct positive integers, none with a prime factor greater than 26, prove that M contains four distinct elements whose product is the fourth power of an integer. [Compare to Optional exercise #4.]

8. Find the set of all positive integers n with the property that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.